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FOUNDATIONS

*Post, Emil L. **The Two-Valued Iterative Systems of Mathematical Logic.** Annals of Mathematics Studies, no. 5. Princeton University Press, Princeton, N. J., 1941. viii+122 pp. $1.75.

The paper was presented before the American Mathematical Society in 1920 [Abstracts in Bull. Amer. Math. Soc. 26, 437 (1920)]. By a two-valued truth-function, we may understand simply a function, of which the independent variables range over a domain of two objects, and of which the value of the dependent variable for each set of arguments is taken from the same domain. For a given set of n distinct variables as the independent variables, there are exactly 2^{2^n} such functions, each of them being describable by a truth-table which gives the function value for each of the 2^n sets of arguments. In the truth-table method, the primitives of the propositional calculus are interpreted by truth-functions; in other words, truth-tables are assigned to them. The assigned functions determine a class of functions which are obtainable from them by repeated use of substitution, that is, by explicit definition, or in Post's terminology by iterative processes: If $f(x_{i_1}, \cdots, x_{i_n})$ is a given function of distinct variables $x_{i_1}, \cdots, x_{i_n}$, and $X_1, \cdots, X_n$ are variables and given functions, which in their totality depend on the distinct variables $x_{j_1}, \cdots, x_{j_m}$, then $f(X_1, \cdots, X_n)$ is an iterative process yielding a function of the latter. The functions of this class are the functions which can be represented by formulas of the calculus, on the basis of the assignment of truth-tables to the primitives. The class of these functions is called the system of functions generated by the assigned functions as generators. It is an iteratively closed system of functions, that is, a non-vacuous class of functions closed under substitution (for any iterative process with given functions in the class, the resulting function is in the class). For the propositional calculus of Principia Mathematica, with the primitives "negation" and "disjunction" and the truth-tables which their meanings suggest (which calculus, considered independently of the particular selection of primitives, we now term the classical calculus of propositions), the generated system is the complete system C_1 of all two-valued truth-functions. In general, given a finite list of two-valued truth-functions as generators, we shall obtain as generated iterative system a subsystem of C_1.

For the present paper, Post posed two problems. (1) What sets of generators give the complete system C_1 (or what primitives can be taken for the classical calculus of propositions)? (2) For arbitrary generators, what are the systems generated (or what are the non-equivalent sub-languages of the classical calculus of propositions)? These can be considered as problems of pure mathematics, without referring to their application to two-valued propositional calculi. The definition of an iteratively closed system of two-valued truth-functions does not depend on the assumption that there is a finite set of generators. From Post's solution of the second of the problems, there appeared the remarkable result that every iteratively closed system of two-valued truth-functions can be generated by a finite set of generators. The paper is oriented about the notion of an (iteratively) closed system (of two-valued truth-functions). The number of distinct independent variables of a function is called its order; the order of a set of functions is the greatest order of any function of the set; and the order of a closed system is the least order of any set of generators of the system. Post catalogues the distinct closed systems completely, as follows. There are 9 systems of order one, 37 of order two, 20 of order three, and 8 of each higher order. The cataloguing is accomplished by means of iterative conditions, that is, properties of functions such that, whenever the given functions for an iterative process have the property, so must the resulting function. To begin with, it is observed that a first order set of generators generates a system comprising only first order functions; and examination of the 15 non-vacuous subsets of the 4 first order functions of a given variable reveals the 9 first order systems. From any function a first order function is obtained whenever we substitute some one variable for all the independent variables; and for any closed system the functions so obtainable must constitute a closed first order system, called the associated first order system. Thus all closed systems are separated into 9 categories. By imposing on the functions further iterative conditions, such as self-duality, in various combinations, the catalog is eventually completed. Eight of the third order systems, together with the systems of higher order, are obtained as eight infinite families of systems F_i^μ ($i=1, \cdots, 8$; $\mu=2, 3, \cdots$), where F_i^μ is of order $\mu+1$ and the parameter μ appears in the iterative conditions. All iterative conditions employed are effectively applicable, so that, for any given function and any named one of the systems, one can ascertain whether or not the function belongs to the system.

In the presentation, the author follows the suggestion of the referee of the original version of the paper [l.c.] that the truth-tables be replaced by logical expressions in the Jevons notation (1864). In this a function is represented by expansions, each of which consists of a pair of expressions, of which the first is essentially (in the terminology of Hilbert-Ackermann) a disjunctive normal form of the function (with 0 and 1 also allowed), and the second is the like for the negation of the function. With the aim of combining the precision of his own truth-table method with somewhat the logical context of Jevons, Post treats the functions in this connection as membership functions of classes; for such a function $f(B, D, \cdots, H)$, the independent variables $B, D, \cdots, H$ range over the subclasses of a given non-vacuous class (the latter being called the universal class), and each function value $f(B, D, \cdots, H)$ is such a class, subject to the general condition on the function that the membership or non-membership of an arbitrary member x of the universal

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class in the class $f(B, D, \cdots, H)$ should depend only on the membership or non-membership of x in each of the classes $B, D, \cdots, H$. The membership functions of classes are abstractly isomorphic with the two-valued truth-functions. The iterative conditions employed are mostly stated as formal properties of the expansions. From an expansion of a function, one can obtain its truth table, and vice versa; and the conditions translate readily into terms of the truth-tables. The relationships of inclusion among the closed systems are presented graphically by means of an inclusion diagram, with dual symmetry. From this diagram, one can extract much specific information about the closed systems. To solve the first of the original two problems, Post gives (a) necessary and sufficient conditions that a set of functions generate the complete system, (b) a classification of such sets subject to the requirement of independence, and (c) a catalog of 36 essentially distinct such sets which is complete when a further irredundancy condition is imposed.

S. C. Kleene (Amherst, Mass.).

***Gonseth, F. Sur la doctrine préalable des vérités élémentaires.** Les entretiens de Zurich sur les fondements et la méthode des sciences mathématiques (6–9 Décembre, 1938), 1941, 13–24.

This is a general introduction to the conference on mathematical foundations at Zurich in 1938. The author first points out that there are certain philosophical presuppositions at the basis of formalism and intuitionism alike; and the antiphilosophical attitude of proponents of those schools must be taken as a rejection, not of philosophy in general, but of types of philosophy not in accord with their views. Indeed, he says every approach to the foundations of mathematics starts from a "doctrine préalable," more or less explicit, concerning the nature of thought and knowledge. The explicit statement of such a doctrine, not as a fixed dogma, but as subject to change like any scientific theory adequate to the needs of the present day, is, as he conceives it, the principal problem of mathematical foundations; and he hopes that the congress will make progress toward that end.

H. B. Curry (State College, Pa.).

***Skolem, Th. Sur la portée du théorème de Löwenheim-Skolem.** Les entretiens de Zurich sur les fondements et la méthode des sciences mathématiques (6–9 Décembre, 1938), 1941, 25–47, discussion, 47–52.

This is a general account of the theorem mentioned in title and its significance. After a statement of the theorem and a reference to the original proof, the author sketches his own generalizations and improvements in the proof, with incidental mention of the theorem of Herbrand. He then shows that systems of axiomatic set theory can be so formulated that the theorem or its generalization applies; in particular, he shows this for the von Neumann and Zermelo-Fraenkel theories by giving the formulations explicitly. This situation is commonly known as Skolem's paradox; for it shows (non-constructively, of course) that these systems, if consistent at all, have an enumerable model despite the fact that they contain cardinals of very high power. The author discusses the significance of this at some length; he argues that it is not paradoxical; that it does not disclose a fundamental weakness of the axiomatic method; that it merely shows that concepts such as denumerability and finiteness are relative, indeed the absolutely non-denumerable does not exist; in short, it shows that these systems cannot be categorical. The paper concludes with a section in which the author discusses the non-categoricalness of arithmetic, particularly his own direct proof that the axioms for arithmetic have models differing from the usual one.

H. B. Curry (State College, Pa.).

***Fréchet, M. L'analyse générale et la question des fondements.** Les entretiens de Zurich sur les fondements et la méthode des sciences mathématiques (6–9 Décembre, 1938), 1941, 53–73, discussion, 73–81.

This paper contains two parts, the first relating to mathematics in general, the second to general analysis. The thesis of the first part is that mathematics is not a purely deductive science. Every mathematical theory involves four stages: first, inductive synthesis from experience, second, axiomatic formulation, third, deductive development from the axioms, and fourth, experimental verification of the conclusions. Thus our fundamental notions of geometry and analysis, as well as those of number, are derived by abstraction and idealization from experience. An axiomatic system without such a relation to experience would be a pure play-thing, not mathematics. Of course the relation may be indirect; and thus there may be a secondary motivation arising through generalization and unification. This leads to the second part of the paper in which some of the most elementary facts about general analysis are stated. [Reviewer's note: those who say that mathematics is purely deductive do not generally deny anything that Fréchet says, but simply confine the term "mathematics" to the second and third stages; this, as Fréchet himself says, is a purely verbal question.] The discussion amplifies and criticizes some points made in the paper.

H. B. Curry.

***Łukasiewicz, Jan. Die Logik und das Grundlagenproblem.** Les entretiens de Zurich sur les fondements et la méthode des sciences mathématiques (6–9 Décembre, 1938), 1941, 82–100, discussion, 100–108.

This is a general survey of the calculus of propositions. The paper presupposes acquaintance with the author's notation without parentheses, but otherwise avoids technicalities. First the author discusses the relation between classical algebra and the intuitionistic. He gives a set of axioms for the classical algebra which are the same as those of Bernays except for the axioms for negation. These are: (1) *CCpNqCqCNp* [last letter is misprinted "*q*"]; (2) *CNpCpq*; (3) *CCCpNpqCCpqq*. From these one gets the Heyting algebra by omitting the last axiom, and the "minimal" algebra of Johansson by omitting the last two. Next the author discusses the matrix method and mentions a theorem of Lindenbaum to the effect that every system of formulas of the propositional algebra is representable by a matrix with at most enumerably many elements. For the Heyting algebra Gödel showed this matrix is infinite; and an infinite matrix representation was actually given by Jaśkowski. The rest of the paper concerns three valued systems. There are two similar but distinct systems, one proposed by Heyting and one by Łukasiewicz himself; these differ intuitively in the meaning assigned to the middle value. The Heyting system is intermediate between the intuitionistic algebra and the classical one. As for the Łukasiewicz system, it has been axiomatized by Wajsberg. Later Słupecki showed that, by adjoining to C and N a new function T, and adding appropriate axioms, a system is obtained which is complete in both of the following senses: (1) every function definable in the three valued matrix is representable in the system, and (2) if any formula

not deducible in the system is adjoined to it as new axiom, the new system is inconsistent. This shows that the system is not a subsystem of the two valued system but is incompatible with it. The author claims that it can represent intuitive modal reasoning, but this point is not clear. The discussion at the end of the paper has to do with this intuitive representation. There are no references to the original papers. *H. B. Curry* (State College, Pa.).

*Lebesgue, H. **Les controverses sur la théorie des ensembles et la question des fondements.** Les entretiens de Zurich sur les fondements et la méthode des sciences mathématiques (6–9 Décembre, 1938), 1941, 109–122, discussion, 122–124.

The author outlines a philosophy of mathematics which may well be called heuristic. This is a philosophy created by mathematicians themselves, and is based on an examination of the methods of mathematics as it actually exists. Such an examination he says will certainly lead beyond purely logical considerations. What makes a proof convincing is not logic, but its concordance with a gradually evolved intuition of a subjective character. This is the source of mathematical certainty; a certainty which is never absolute. In consequence mathematics cannot be separated entirely from psychological and aesthetic considerations. The author discusses the axiom of choice and other questions of Cantorian set-theory from such a point of view. *H. B. Curry* (State College, Pa.).

*Sierpinski, Waclaw. **L'axiome du choix et l'hypothèse du continu.** Les entretiens de Zurich sur les fondements et la méthode des sciences mathématiques (6–9 Décembre, 1938), 1941, 125–134, discussion, 134–143.

This is a clear and interesting, if somewhat elementary, expository account of the axiom of choice, the continuum hypothesis and related principles. The significance of these principles is brought out by stating a judicious selection of other principles which imply them, are equivalent to them or are consequences of them. The paper is followed by a long discussion, participated in by various persons, of a more philosophical nature; part of this pertains to the preceding paper. *H. B. Curry* (State College, Pa.).

*Bernays, P. **Sur les questions méthodologiques actuelles de la théorie hilbertienne de la démonstration.** Les entretiens de Zurich sur les fondements et la méthode des sciences mathématiques (6–9 Décembre, 1938), 1941, 144–152, discussion, 153–161.

According to the Gödel theorem the consistency of formal systems of arithmetic and analysis cannot be demonstrated by methods which can be formalized within the systems themselves; consequently it is necessary to use methods which transcend Hilbert's original conception of a finite proof. This paper is concerned with a non-technical discussion of the nature and significance of such methods. There are two of these from which a proof of the consistency of arithmetic has been obtained; namely, the method of intuitionism [consistency proof obtained by Gödel, Ergebnisse math. Kolloqu. Wien **4**, 34–40 (1933)] and that of Gentzen [Forschungen für Logik, Heft 4, Leipzig, 1938]. In regard to intuitionism the author points out that it has a non-constructive element in the admission of arbitrary propositions as antecedents of an implication. In regard to the Gentzen method, which is based on an intuitive form of the principle of transfinite induction for ordinals less than the first ϵ-number, the author explains the fundamental principle and shows it has a character intermediate between the intuitionistic method and that of Hilbert. In conclusion the author advances reasons for believing that it may be possible eventually to obtain a consistency proof for analysis by using a generalization of the Gentzen method. The discussion at the conclusion of the paper relates principally to the system of set theory proposed by Finsler, and sheds some light on the philosophy back of that system. *H. B. Curry* (State College, Pa.).

*Finsler, P. **A propos de la discussion sur les fondements des mathématiques.** Les entretiens de Zurich sur les fondements et la méthode des sciences mathématiques (6–9 Décembre, 1938), 1941, 162–180, discussion, 181–187.

This is a resume of the author's views on the foundations of the theory of sets, particularly the axiom of choice. These ideas were originally expounded in two papers in Math. Z. **25**, 676–713 (1926). The present address adds very little to the content of those papers; but in the discussion and supplementary notes to this and the preceding articles the philosophy back of this system is gone into in some detail. The author takes a position of extreme Platonism; he postulates an absolute logic whose nature is not clear except that it includes the dizziest applications of the law of excluded middle, the absoluteness of notions such as cardinal number, the existence of objects which cannot be defined in any finite fashion, etc. *H. B. Curry* (State College, Pa.).

*Gonseth, F. **Sur le rôle unificateur de l'idée de dialectique.** Les entretiens de Zurich sur les fondements et la méthode des sciences mathématiques (6–9 Décembre, 1938), 1941, 188–209.

In this, the concluding paper of the conference, the author sums up the work of his predecessors. He points out that the differences of opinion are as great as before. He attempts to find a common basis in the notion of a "dialectic," that is, "une argumentation essentiellement informée par trois moments principaux: par la signification des choses dont on parle, par les fins que la pensée poursuit en en parlant, et enfin par les façons que l'on a déjà découvertes d'en parler avec bon sens et efficacité." He discusses the relationship of this notion to the ideas of certain of the preceding speakers and to formalism. The paper is of interest primarily to those concerned with more philosophical aspects of mathematical methodology. *H. B. Curry* (State College, Pa.).

Levi, Beppo. The notion of "deductive domain" as an element of orientation on the questions of foundations of mathematical theories. Publ. Inst. Mat. Univ. Nac. Litoral **2**, 179–208 (1940). (Spanish) [MF 4233]

This is an amplified exposition of the author's views in regard to the foundations of mathematics, which he has developed in a series of papers extending back to 1908. [The most accessible are Math. Ann. **90**, 164–173 (1923); Fund. Math. **23**, 63–74 (1934); cf. the review of this last paper by Ackermann in Jahrb. Fortschr. Math.] The fundamental thesis is that the notion of aggregate is basic for mathematics; that in every mathematical theory a certain finite number of such aggregates are presupposed, from which it is postulated that we can choose arbitrary (variable) elements by a single indecomposable "act of thought"; these aggregates, which are called "prime aggregates," define the "deductive domain" within which the theory develops. Such a prime aggregate is, for example, the set

of natural numbers; this, the author argues, is not obtainable either from logic or from experience, but is a primitive intuition which the mind conceives at a certain stage of development. Likewise the real numbers form a prime aggregate; it cannot be defined in terms of integers but is a "primitive idea," that is, a simple intuition. Some of the more involved theorems of analysis require three or more such prime aggregates; for example, the aggregate of real functions may be adjoined as a new primitive idea. Certain anomalies in regard to these theorems, the author claims, can be explained by considering the nature of the deductive domain within which they are valid; and the author states a principle, called the principle of approximation (which dates back at least to 1918), regarding the formation of more extended prime aggregates. On the other hand arguments which essentially involve the axiom of choice are rejected. As to the methods of legitimate proof within a given deductive domain, the author leaves this question completely unanswered. The treatment is discursive and obscure, so that the reviewer cannot be sure he has understood it; but it is clear that this is an idealistic, not a formalized, theory. *H. B. Curry* (State College, Pa.).

Curry, Haskell B. Some aspects of the problem of mathematical rigor. Bull. Amer. Math. Soc. **47**, 221–241 (1941). [MF 4166]

This paper constitutes the substance of an invited expository address delivered to mathematicians. Its purpose is to explain the fundamental concepts of formalism and to add new suggestions and criticisms in matters of detail. The chief aspects presented concern (i) a critique of non-formal theories, (ii) the notion of a formal system (illustrated by Dickson's postulates for groups and an abstract system G), (iii) the notion of a calculus, (iv) metatheory, (v) the definition of mathematics, (vi) acceptability of a formal system (with some pertinent criticisms of extreme demands from intuitionists and formalists). *A. A. Bennett.*

Curry, Haskell B. A formalization of recursive arithmetic. Amer. J. Math. **63**, 263–282 (1941). [MF 4149]

The term "recursive arithmetic" refers to the development of certain parts of the theory of natural numbers by use of elementary logical operations, of the successor function (of a natural number) and of "primitive recursive definitions." The kernel of the idea goes back to Dedekind and Peano and has been developed by many recent writers. In particular, Skolem showed that recursive arithmetic could be formalized without quantifiers. The author here assumes no general underlying logical calculus, but works out from the start an "independent logical system." In addition to a number of non-formal symbols, there are listed primitive terms, rules of formation, axioms, etc. The only primitive predicate of the system is the binary relation of equality between numerical expressions. The author shows how to set up the propositional calculus within the system, and hence the Hilbert-Bernays formalism may be included as an outgrowth of the present scheme. The essential steps are demonstrated in detail. *A. A. Bennett.*

Curry, Haskell B. A revision of the fundamental rules of combinatory logic. J. Symbolic Logic **6**, 41–53 (1941).

This is a formal development of "the ultimate foundations" of combinatory logic as previously dealt with in the author's thesis. The proofs are here given in full. The work is largely a simplification of improvements suggested by some work of Rosser. The method of procedure adopted, instead of setting up a primitive frame and deriving its properties, is to begin with the study of the desired properties, and thence to arrive at a possible basis. In particular a number of important characteristics of certain types of relations are examined with regard to their interdependence. The study of properties of equality leads finally to a theorem establishing an available definition for equality. The four combinators B, C, W, K appearing in the axiom schemes are finally shown to be reducible to two, namely, S and K (following Schönfinkel). *A. A. Bennett.*

Curry, Haskell B. Consistency and completeness of the theory of combinators. J. Symbolic Logic **6**, 54–61 (1941).

Making use of the discussion in the paper reviewed above, the author extends to the whole theory of combinators the proof for consistency applied in his thesis to the combinators B, C, W, K. A key theorem in this line of investigation is due to Church and Rosen: "If $\mathfrak{M}$ is convertible to $\mathfrak{N}$, then there can be found (constructively) a λ-term $\mathfrak{P}$ such that $\mathfrak{M}$ and $\mathfrak{N}$ reduce to $\mathfrak{P}$." The consistency theorem and completeness theorems are demonstrated. "Every term $\mathfrak{A}$ asserted in $\mathfrak{K}$ is of one of the types I, II, or III" (standard forms defined previously). *A. A. Bennett.*

Malisoff, William Marias. Meanings in multi-valued logics. Philos. Sci. **8**, 271–274 (1941). [MF 4763]

Blaschke, Wilhelm. Mathematik und Leben. Hamburg Math. Einzelschr. **27**, 13 pp. (1940). [MF 4503]

ALGEBRA

Neumann, Hanna. On the elimination rule. J. London Math. Soc. **15**, 281–293 (1940). [MF 4039]

The word elimination, used in the title, refers to chess matches rather than to the familiar algebraic process. A rule often used to determine the winner of two chess teams is as follows: Each game won or drawn by a member of a team counts 1 point, each game lost counts 0, towards the total score of the team. The team with the higher score wins the match, but if both teams have the same score, that team which lost at the last game not drawn wins the match. The problem considered here is that of preassigning positive weights to each game so that the total scores, thus modified, will select the winning team according to the above rule. In general consider a game in which a player may receive any number of points from 0 to m, and let teams X and Y compete in a match of n games. If the score for the ith game is x_i to y_i, then the team Y wins the match if either $x_1+x_2+\cdots+x_n<y_1+y_2+\cdots+y_n$ or $x_1+x_2+\cdots+x_n=y_1+y_2+\cdots+y_n$ and $x_1=y_1$, $x_2=y_2$, $\cdots$, $x_{k-1}=y_{k-1}$, $x_k>y_k$ $(1\leqq k\leqq n)$. The problem is to determine a set of n weights $a_1, a_2, \cdots, a_n$ (independent of the x's, y's or k) so that we have the simpler criterion: The team Y wins if and only if

$$a_1x_1+a_2x_2+\cdots+a_nx_n<a_1y_1+a_2y_2+\cdots+a_ny_n.$$

Such a set of weights is given by $a_i=\gamma_n-\gamma_{n-i}$, where $\gamma_0=0$, $\gamma_1=1$, $\gamma_2=m+1$, $\gamma_3=m^2+m+2$, $\cdots$, $\gamma_r=(m+1)\gamma_{r-1}-(m-1)\gamma_{[(r-1)/2]}-\gamma_{[(r-2)/2]}$. This choice gives weights as small as possible. *D. H. Lehmer* (Berkeley, Calif.).

Gupta, Hansraj. A problem in combinations. Math. Student **8**, 131–132 (1940). [MF 4351]

The problem under consideration is whether it is possible to select, from m soldiers, guards for m days in such a way that any two soldiers are companions on two days only. If the guard consists of n soldiers each day the author states that no solution exists unless $m=1+n(n-1)/2$. This condition, however, is not necessary if $n=2$, although it may be necessary for $n>2$. The author exhibits solutions in which $(n, m)=(3, 4)$, $(4, 7)$, $(5, 11)$ and $(6, 16)$ and conjectures that there is a solution for every value of n.

D. H. Lehmer (Berkeley, Calif.).

van der Corput, J. G. Symmetric functions. Chr. Huygens **18**, 251–277 (1940). (Dutch) [MF 4359]

Turán, Pál. On extremal problems concerning determinants. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **59**, 95–105 (1940). (Hungarian. English summary) [MF 4448]

Let us consider a determinant of order n whose elements are -1 and $+1$. The problem of determining the maximum M_n of these determinants has been considered by several authors. It has been conjectured that, if $n=4m$, then $M_n=n^{n/2}$ (that is, if $n=4m$, there exists an orthogonal determinant whose elements are -1 and $+1$). This conjecture has been proved only for special values of n. The author considers the following problem: He prescribes arbitrarily the elements of the first k rows ($k<n$) and denotes them by $\epsilon_{\mu,\nu}$ ($\mu=1, 2, \cdots, k$; $\nu=1, 2, \cdots, n$) (clearly we can suppose that no two rows are essentially identical). Denote the maximum of such determinants by $M_n(k, \epsilon)$. Then $M_n(k, \epsilon)\geqq((n-k)!)^{\frac{1}{2}}D^{\frac{1}{2}}$, where D denotes the determinant of order k, which we obtain by composing the matrices

$$\begin{pmatrix}\epsilon_{11} & \cdots & \epsilon_{1n}\\ \cdots & \cdots & \cdots\\ \epsilon_{k1} & \cdots & \epsilon_{kn}\end{pmatrix},\quad \begin{pmatrix}\epsilon_{11} & \cdots & \epsilon_{1n}\\ \cdots & \cdots & \cdots\\ \epsilon_{k1} & \cdots & \epsilon_{kn}\end{pmatrix}$$

row by row. Also

$$M_n(2, \epsilon)\geqq 2((n-1)!)^{\frac{1}{2}},\quad M_n(3, \epsilon)\geqq 4((n-2)!)^{\frac{1}{2}}.$$

By the theorem of Hadamard it is of course clear that $M_n\leqq n^{n/2}$.

P. Erdös (Philadelphia, Pa.).

Dines, Lloyd L. On the mapping of quadratic forms. Bull. Amer. Math. Soc. **47**, 494–498 (1941). [MF 4544]

Let $P(z)$ and $Q(z)$ be real quadratic forms in the real variables $z^1, z^2, \cdots, z^n$, and let $\mathfrak{M}$ be the set of points in the xy-plane into which the z-space is mapped by the transformation $x=P(z)$, $y=Q(z)$. The author shows that $\mathfrak{M}$ is a convex set and, if furthermore $P(z)$ and $Q(z)$ have no common zero except $z=(0)$, then $\mathfrak{M}$ is closed as well as convex, and is in fact either the entire xy-plane or an angular sector of angle less than π. It follows that necessary and sufficient conditions that there exist real λ, μ such that, for all $z\neq(0)$, $\lambda P(z)+\mu Q(z)>0$ are that: (1) there exist real a, b such that the two equations $P(z)=a$, $Q(z)=b$ are inconsistent for real z; and (2) $P(z)$ and $Q(z)$ have no common zero except $z=(0)$. The first condition alone is both necessary and sufficient that there exist real λ, μ such that, for all real z, $\lambda P(z)+\mu Q(z)\geqq 0$. These results are closely related to previous work of Finsler [Comment. Math. Helv. **9**, 188–192 (1937)], Albert [Bull. Amer. Math. Soc. **44**, 250–253 (1938)], Reid [Bull. Amer. Math. Soc. **44**, 437–440 (1938)] and Hestenes and McShane [Trans. Amer. Math. Soc. **47**, 501–512 (1940); cf. these Rev. **2**, 119].

N. H. McCoy (Northampton, Mass.).

Equations, Polynomials

Sebastião e Silva, José. Problems concerning rational functions of the roots of an algebraic equation. Portugaliae Math. **2**, 20–35 (1941). (Portuguese) [MF 4501]

In section 5 of this paper the author outlines a procedure to find the irreducible factors of a polynomial in a field. The coefficients of a normalized factor belong to a field generated by the symmetric functions of p of the n roots. In favorable cases, this field is generated by the sum or the product of these p roots. Determine the equation whose roots are all sums of p roots; if there is a factor in the field, then this equation has a root in the field. Assuming that it is possible to find all roots of an equation in a field, the preceding section furnishes a method of finding the coefficients of the factor, if it exists. The method, according to the author, does not differ basically from the known ones. This theory is preceded by considerations from Galois theory (classical form) and followed by remarks on resolvents, etc. In one place of the paper [end of section 2] and in the appended notes the language of modern abstract algebra is used.

M. A. Zorn (Los Angeles, Calif.).

Montgomery, John C. The roots of a polynomial and its derivative. Bull. Amer. Math. Soc. **47**, 621–624 (1941). [MF 5056]

In this article the following identity is established between the n zeros $\alpha_1, \alpha_2, \cdots, \alpha_n$ of an nth degree polynomial $f(z)$ and any q distinct zeros $\beta_1, \beta_2, \cdots, \beta_q$ ($\beta_j\neq\alpha_k$) of the derivative $f'(z)$:

$$\sum_{j=1}^{n}((\beta_1-\alpha_j)(\beta_2-\alpha_j)\cdots(\beta_q-\alpha_j))^{-1}=0.$$

As the author states, this identity is suggested by and derivable from one given by M. Marden [Trans. Amer. Math. Soc. **45**, 355–368 (1939)]. The identity is used to prove, when $f'(z)=(z^q-1)g(z)$, that the curve $\rho^q\cos q\theta=1$ either passes through all the zeros of $f(z)$ or separates them, and that the lines $\arg z=k\pi/q$, $k=0, 1, \cdots, 2q-1$, pass through all the zeros of $f(z)$ or at least one zero of $f(z)$ lies in a sector $k\pi/q\leqq\arg z\leqq\pi(k+1)/q$ with k odd and at least one zero in such a sector with k even.

M. Marden.

v. Szökefalvi Nagy, Gyula. Über Polynome mit lauter reellen Nullstellen. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **59**, 72–94 (1940). (Hungarian. German summary) [MF 4447]

The author, generalizing his previous theorem, proves the following result: Let $f(x)$ be a polynomial of degree n, $n=m+m_1+m_2$; $m>0$, $m_1>0$, $m_2>0$, whose roots $x_1\leqq x_2\leqq\cdots\leqq x_n$ are all real. Denote by $y_1, y_2, \cdots, y_{n-1}$ the roots of $f'(x)$. Put $d_m=x_{n-m_2}-x_{m_1}$; then we have, if $\mu=\max(m_1, m_2)$,

$$y_{n-m_2-1}-y_{m_1}=d_m'\geqq(2/(\mu+1))(m\mu/n)^{\frac{1}{2}}d_m.$$

The proof is based on the following simple remark: Let $f(x)=(x-\alpha)f_1(x)$; the roots of $f(x)$ are real; let us increase α by s and leave the roots of $f_1(x)$ unchanged; then those roots of $f'(x)$ which are different from the roots of $f_1(x)$ also increase, but by an amount less than s.

P. Erdös.

v. Sz. Nagy, Gyula. Über die reellen Nullstellen gewisser Polynome mit Parametern. Acta Univ. Szeged. Sect. Sci. Math. **10**, 36–41 (1941). [MF 4429]

In this note, the zeros of the function $\phi(x)=g(x)F(x)-f(x)G(x)$ are discussed when $f(x)$, $g(x)$, $F(x)$ and $G(x)$ are real polynomials. An immediate result is that, if between two real zeros a_1 and a_2 of $f(x)$ lies an odd number of zeros of $g(x)$ and an even number of $F(x)$, then on the closed interval (a_1, a_2) lies at least one zero of $\phi(x)$. From this result is derived a number of theorems, similar to the following: If $f(x)=A(x-a_1)(x-a_2)\cdots(x-a_m)$ and $g(x)=B(x-b_1)(x-b_2)\cdots(x-b_{m-1})$ with $a_1<b_1<a_2<\cdots<b_{m-1}<a_m$, and if $p<m$ intervals (a_k, a_{k+1}) contain an odd number of zeros of $F(x)$ and $q<m-1$ intervals (b_k, b_{k+1}) contain an odd number of zeros of $G(x)$, then $\phi(x)$ has at least a number $r=\max(m-1-p, m-2-q)$ of real zeros. These theorems give a lower bound on the number of intersection points of the curves $y=f(x)/g(x)$ and $y=F(x)/G(x)$.

M. Marden (Milwaukee, Wis.).

Mirakyan, G. Verallgemeinung der statischen Interpretation von Stieltjes der Nullstellen einiger Polynome. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **16**, 158–166 (1940). (Russian. German summary) [MF 4743]

The author extends the known result of Stieltjes concerning the extremal (statical) property of the zeros of Jacobi polynomials to a system of points in the plane. We wish to find the position of equilibrium in the plane of n mass-points (mass of each equals 1), being repulsed from $p(\geqq 3)$ fixed mass-points according to the law of logarithmic potential. The method is that of Stieltjes: setting up a linear differential equation of second order and studying (sketchily) its polynomial solutions. The first part of the paper, dealing with Laguerre and Hermite polynomials, does not contain anything essentially new. [Here reference should have been made to J. Shohat, Mémor. Sci. Math., fasc. **66**, pp. 19–20; Rec. Math. [Mat. Zbornik] N.S. **4 (46)**, 357–369 (1938). Concerning a mechanical interpretation of the zeros of Laguerre polynomials, cf. H. Bateman, Amer. Math. Monthly **38**, 1–8 (1931).] *J. A. Shohat.*

Ostrowski, Alexandre. Recherches sur la méthode de Graeffe et les zéros des polynomes et des séries de Laurent. Chapitres III et IV. Acta Math. **72**, 157–257 (1940). [MF 3804]

[The first two chapters of this memoir were published in the same vol., 99–155 and were reviewed in these Rev. **1**, 323.] The problem of deriving information about the roots of a polynomial $f(z)$ (and more generally of a Laurent series) from the absolute values of its coefficients is now investigated in terms of the deviations $D_\nu=R_{\nu+1}/R_\nu=\exp(\kappa_{\nu+1}+\kappa_{\nu-1}-2\kappa_\nu)(\geqq 1)$ of the Newton diagram of the Laurent series $f(z)=\sum_{-\infty}^{\infty}a_\nu z^\nu$. The second fundamental theorem allows the isolation in a ring of a simple root of the equation $f(z)=0$, provided two consecutive deviations are sufficiently large (e.g., greater than 7). If a single deviation D is greater than 9, a zero-free ring is indicated. If (1) $D_m\geqq 9$, $D_n\geqq 9$, $(m<n)$ a ring containing exactly $n-m$ zeros, as well as an estimation of their product, is given by means of Jensen's formula. Of particular significance for Graeffe's method is the following result: If $0<|\zeta_1|\leqq|\zeta_2|\leqq\cdots\leqq|\zeta_n|$ are the absolute values of the roots of an equation, then $D_m>9$ is sufficient to insure that $|\zeta_m|<|\zeta_{m+1}|$; moreover, if $|\zeta|_m<|\zeta_{m+1}|$ then the deviations D_m of the Graeffe transforms $f_k(z)$ will satisfy the condition $D_m>9$ for sufficiently large k. A study of the Newton diagram of the product of two Laurent series culminates through a sequence of lemmas in the third fundamental theorem. Very roughly its purpose is the study of the relation between the Laurent series $f(z)$ and the polynomial $Q(z)$ having as zeros the $n-m$ zeros of $f(z)$ isolated under the assumption (1). This theorem with its corollaries is of significance for the second major problem (B) concerning Graeffe's method: the determination of the arguments of the zeros. The fourth fundamental theorem concerning the relative continuity of the zeros of a polynomial was published independently [C. R. Acad. Sci. Paris **209**, 777–779 (1939); these Rev. **1**, 97]. It is of significance for the third problem (C) [these Rev. **1**, 323]. The last 27 pages are devoted to the applications of the general results to the theory and practice of Graeffe's method. They include a critical analysis of devices due to Graeffe, Encke, Carvallo and Runge. Three significant numerical examples are treated in detail. *I. J. Schoenberg.*

Dieudonné, J. Quelques résultats quantitatifs de théorie analytique des polynomes. J. Math. Pures Appl. (9) **19**, 121–132 (1940). [MF 4618]

The author deals with the problem of determining the smallest circle $|z|\leqq r=r(q, n, m, p)$ of the following property. Each function of the form $\sum a_\nu(z-z_\nu)^{-m-1}$, $1\leqq\nu\leqq n$, where $m\geqq 1$, $|z_1|\leqq 1$, $\cdots$, $|z_q|\leqq 1$, possesses at least p roots in $|z|\leqq r$. In a previous paper [Ann. École Norm. (3) **54**, 101–150 (1937), in particular p. 146] he showed that such a circle exists if $p=N=m(q-1)$. In the present paper $1\leqq p\leqq N$ is assumed and the following results are obtained. First

$$r(n, n, m, m(n-1))\leqq(1-2^{-\lambda})^{-1}, \qquad \lambda n(n+m)=1.$$

Further $r(q, n, m, 1)\leqq\gamma(n-q+1)(n-1)(q-1)^{-1}$, $2\leqq q\leqq n$. [A previous result [C. R. Acad. Sci. Paris **203**, 767–769 (1936)] states that $r(n, n, m, 1)=1$.] Finally a related special problem is discussed with the precise determination of the corresponding radius. *G. Szegö.*

Carlitz, L. An analogue of the Bernoulli polynomials. Duke Math. J. **8**, 405–412 (1941). [MF 4604]

The author defines a set of polynomials with coefficients in $GF(p^n, x)$ that are analogous to Bernoulli polynomials, just as he has defined analogues of Bernoulli numbers in earlier papers. This paper contains properties of these polynomials, the main one being an analogue of the von Staudt theorem on Bernoulli numbers. *H. W. Brinkmann.*

Abstract Algebra

Nakamura, Masahiro. Closure in general lattices. Proc. Imp. Acad. Tokyo **17**, 5–6 (1941). [MF 4331]

The author defines a "closure operator" α on a complete lattice L by the rules: $a\alpha\geqq a$, $(a\cup b)\alpha=a\alpha\cup b\alpha$, $(a\alpha)\alpha=a$, $0\alpha=0$. He establishes a dual isomorphism between closure operators on L and the meet-complete sublattices of "closed" elements of L containing 0 and I. The lattice operations within these resulting lattices generalize the "combinations of topologies" defined by the reviewer. *G. Birkhoff.*

Kurosch, A. G. The theorem of Jordan-Hölder in arbitrary structures. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 110–116. (Russian) [MF 3509]

An analogue of the Jordan-Hölder-Schreier theorem is formulated and proved for arbitrary structures (lattices). This result coincides with that obtained by O. Ore [Trans. Amer. Math. Soc. **41**, 266–275 (1937), Theorem VII]. The methods used in the present paper differ in many details from those of Ore. The notion of normal elements of a structure (seminormal in Ore's terminology) is formulated as follows: An element a is called normal in the structure S if the relations $b \leqq b' \leqq a+b$, $a \leqq c \leqq c' \leqq a+b$, $bc=b'c'$ imply $b=b'$, $c=c'$ (where b, b', c, c' are arbitrary elements of S). This formulation seems to be particularly convenient for a simple proof of the generalized Jordan-Hölder theorem.

W. Hurewicz (Chapel Hill, N. C.).

Kawada, Yukiyosi. Über die homomorphe Darstellung der Verbände und der multiplikativen Systeme. Proc. Imp. Acad. Tokyo **16**, 537–542 (1940). [MF 4282]

The author considers first the congruence relations on a distributive lattice L, and also on any multiplicative system M (algebra with a single idempotent, commutative and associative operation). He shows that there are strong analogies between the two cases, and relates the analogy with the embedding (by ideals) of M in a distributive lattice. Results of MacNeille and Clifford are used.

G. Birkhoff (Cambridge, Mass.).

Vaidyanathaswamy, R. The ideal-theory of the partially ordered set. Proc. Indian Acad. Sci., Sect. A. **13**, 94–99 (1941). [MF 4374]

The author gives a proof of MacNeille's theorem on the completability of any partially ordered set; his proof involves the notion of "comprincipal" ideals (intersections of principal ideals); the concept of comprincipal ideal as such is new. *G. Birkhoff* (Cambridge, Mass.).

Dilworth, R. P. Ideals in Birkhoff lattices. Trans. Amer. Math. Soc. **49**, 325–353 (1941). [MF 4336]

The author first defines upper semi-modular ("Birkhoff") lattices L in general. He requires that, in the dual of L, if a principal ideal (a) covers an ideal B, and if C is an ideal contained in (a) but not in B, then C covers $B \cap C$. The power of this condition depends on the fact that every principal ideal has maximal subideals. It is stronger, in the absence of the descending chain condition, than the "point-free exchange axiom" proposed by MacLane. The author then shows that, in a lattice L satisfying the ascending chain condition, every element is uniquely expressible as a reduced meet of irreducibles if and only if MacLane's axiom holds and another simple condition holds. This generalizes earlier results of Dilworth. Finally, if L is a "Birkhoff lattice," the number of components (but not the individual components) is unique if and only if the ideals covered by any principal ideal generate a convex modular sublattice. This generalizes the Kurosch-Ore theorem. *G. Birkhoff.*

Mori, Shinziro. Über die Produktzerlegung der Hauptideale. IV. J. Sci. Hirosima Univ. Ser. A. **11**, 7–14 (1941). [MF 4643]

Let R be a commutative ring without a unity element for multiplication. The author proves that each principal ideal of R is a power product of a finite number of prime ideals if and only if $R=M+K$ or $R=M$, where K is a field and M a ring without unity such that (i) each $a \neq 0$ in M is not the product of infinitely many elements, (ii) the principal ideal (p) of each indecomposable element $p \in M$ is the whole ring M. The field K does not occur in above direct decomposition if R does not contain any proper divisors of zero.

O. F. G. Schilling (Chicago, Ill.).

Matusita, Kameo. Nachtrag zu meiner Note "Ueber die Idealtheorie im Integritätsbereich mit dem eingeschränkten Vielfachenkettensatz." Proc. Phys.-Math. Soc. Japan (3) **23**, 271–272 (1941). [MF 4767]

Let I be an integral domain which is integrally closed in its quotient field and satisfies the restricted condition for ascending chains of ideals. The author shows that each ideal of I is the unique product of prime ideals. For the proof one has to observe that the first assumption implies the validity of the theory of v-ideals. The second condition implies that each prime ideal is divisorless.

O. F. G. Schilling (Chicago, Ill.).

Kamei, Eiiti. Zum Durchschnittssatze in einartigen Ringen. Proc. Imp. Acad. Tokyo **17**, 95–99 (1941). [MF 4642]

Let $\mathfrak{R}$ be a commutative ring in which each ideal is the intersection of a finite number of primary ideals. Then each chain of ideal quotients $\mathfrak{a} \subset \mathfrak{a} : \mathfrak{a}_1 \subset \mathfrak{a} : \mathfrak{a}_1\mathfrak{a}_2 \subset \cdots$, with $\mathfrak{a}, \mathfrak{a}_i \subseteqq \mathfrak{R}$, is finite. The converse of this statement is proved if, moreover, each proper prime ideal of $\mathfrak{R}$ is divisorless.

O. F. G. Schilling (Chicago, Ill.).

Nakamura, Masahiro. Partially ordered rings. Tôhoku Math. J. **47**, 251–254 (1940). [MF 4025]

The author considers rings R with unity e, which are lattice-ordered abelian groups under addition, in which also (1) for any a, $ne > a$ for some positive integer n, and (2) the product of any two non-negative elements is itself non-negative. He shows that under these circumstances the decompositions of e into disjoint components correspond one-one to the direct decompositions of R, and derives various corollaries from this. *G. Birkhoff.*

Nakano, Hidegorô. Teilweise geordnete Algebra. Proc. Imp. Acad. Tokyo **16**, 437–441 (1940). [MF 4323]

Let $\mathfrak{M}$ be any σ-complete vector lattice. Following Freudenthal, the author defines projection operators in $\mathfrak{M}$, and Stieltjes integrals with respect to $\mathfrak{M}$-valued functions "of bounded variation." A dilatator is then defined as an operator whose domain is "dense" in $\mathfrak{M}$ and which is permutable with every projection operator. The author states that, if $\mathfrak{M}$ has a weak unit, every dilatator can be represented as a Stieltjes integral, and dilatators can be added and multiplied. Furthermore, any infinite series of disjoint ("orthogonal") dilatators is convergent. He also asserts that if $\mathfrak{M}$ is a ring with unity which is a weak unit, under multiplication, then it is isomorphic to the commutative ring of those dilatators whose domain is $\mathfrak{M}$. Finally, extensions to the complex case are sketched. Reference is made to results of Steen; no proofs are given. *G. Birkhoff.*

Richardson, A. R. Algebra of s-dimensions. Proc. London Math. Soc. (2) **47**, 38–59 (1940). [MF 3650]

A set G_s is said to be an algebra of s dimensions if a product is defined in it such that every set of s, but not less than s, elements of G_s belong to G_s. In this paper the case $s=3$ is studied in detail. Various examples of such systems are given, for instance Latin cubes which are ex-

tensions to three dimensions of the ordinary Latin squares. Any ordinary multiplicative system gives rise to systems G_s. One may assume various associative laws in G_s, most derived from the associative laws in a G_3. Some special types of elements, idempotents, zeros, units and other so-called regular elements are studied in some detail. One can obtain representations analogous to the permutation representations of groups. The author determines the conditions for co-set expansions and for the theorem of Lagrange that the order of a subalgebra divides the order of the algebra. Normal subalgebras are defined, they are shown to satisfy the Dedekind axiom and from this fact one obtains immediately the various decomposition theorems for these algebras. *O. Ore* (New Haven, Conn.).

Nakayama, Tadasi. Algebras with anti-isomorphic left and right ideal lattices. Proc. Imp. Acad. Tokyo **17**, 53–56 (1941). [MF 4451]

This paper forms a continuation of the author's previous work on Frobeniusean algebras [Ann. of Math. (2) **40**, 611–633 (1939); **42**, 1–21 (1941); cf. these Rev. **1**, 3 and the following review]. Let A be a ring with a unit element, and assume that the minimum condition for one-sided ideals is satisfied. It is shown that, if the lattice $\mathfrak{A}$ of the left ideals of A and the lattice $\mathfrak{P}$ of the right ideals of A are anti-isomorphic, then the ring A is quasi-Frobeniusean. The l-dimension $d_l(\mathfrak{m})$ of an A-left-module $\mathfrak{m}$ can be defined in the following manner: For a simple $\mathfrak{m}$, the number $d_l(\mathfrak{m})$ is the rank of $\mathfrak{m}$ with regard to the quasi-field of the operator-automorphisms; for an arbitrary $\mathfrak{m}$, we set $d_l(\mathfrak{m})$ equal to the sum of the l-dimensions $d_l(\mathfrak{f}_i)$ of all simple factors $\mathfrak{f}_i$ of a composition series of $\mathfrak{m}$. The r-dimension $d_r(\mathfrak{n})$ of a right module $\mathfrak{n}$ is defined in an analogous manner. If an anti-isomorphism $\mathfrak{l}\to\mathfrak{r}$ between $\mathfrak{A}$ and $\mathfrak{P}$ exists such that (*) $d_l(\mathfrak{l})=d_r(A/\mathfrak{r})$, then A is a Frobeniusean ring. As was previously shown by the author, the lattices $\mathfrak{A}$ and $\mathfrak{P}$ are anti-isomorphic in the case of a quasi-Frobeniusean ring A, and such an anti-isomorphism is given by $\mathfrak{l}\to\mathfrak{r}(\mathfrak{l})$, where the right ideal $\mathfrak{r}(\mathfrak{l})$ consists of all right annihilators of the left ideal $\mathfrak{l}$. In the case of a Frobeniusean ring, $\mathfrak{l}\to\mathfrak{r}(\mathfrak{l})$ satisfies (*). It follows that, if the lattices $\mathfrak{A}$ and $\mathfrak{o}$ are anti-isomorphic, then $\mathfrak{l}\to\mathfrak{r}(\mathfrak{l})$ is an anti-isomorphism; if there exists an anti-isomorphism satisfying (*), then $\mathfrak{l}\to\mathfrak{r}(\mathfrak{l})$ satisfies (*). A generalization of these last results is given for the case of an anti-isomorphism between the lattices of the A-submoduli of a left vector module and a right vector module over A. *R. Brauer* (Chicago, Ill.).

Nakayama, Tadasi. On Frobeniusean algebras. II. Ann. of Math. (2) **42**, 1–21 (1941). [MF 3671]

[The first part appeared in Ann. of Math. (2) **40**, 611–633 (1939); cf. these Rev. **1**, 3.] The author continues his investigations of the successively larger classes of algebras, semi-simple, symmetric, Frobeniusean and quasi-Frobeniusean, begun in Part I of this paper. In Part I the equivalence of representation properties and annihilation properties as definitions of these classes was established. Here it is shown that a Frobeniusean algebra possesses an automorphism which correlates the two kinds of dual correspondences between left and right ideals, the one given by annihilation, the other by representation theory. The properties investigated in Part I are extended from algebras to rings. Also the hereditary character of these properties is investigated for product rings, residue class rings and vector moduli, including a new proof of a theorem of the reviewer. Three results are obtained of particular value for the study of group rings over modular fields (notoriously not semi-simple when the characteristic of the field divides the order of the group), one on Galois moduli, a second on normal bases and a third stating that the ring of a finite group over a quasi-Frobeniusean ring will again be quasi-Frobeniusean. Finally the relation of Köthe's uni-serial rings to Frobeniusean rings is established. *M. Hall* (New Haven, Conn.).

Asano, Keizo and Nakayama, Tadasi. A remark on the arithmetic in a subfield. Proc. Imp. Acad. Tokyo **16**, 529–531 (1940). [MF 4328]

Let K be a finite normal extension of a field k and $\mathfrak{O}$ a subdomain of K in which the usual arithmetic holds (unique factorization of ideals into prime ideals) and such that K is the quotient field of $\mathfrak{O}$. The authors obtain the following criterion: In order that the usual arithmetic hold in $\mathfrak{O}\wedge k=\mathfrak{o}$ it is necessary and sufficient that it hold in $\mathfrak{O}^*=\mathfrak{O}\wedge\mathfrak{O}'\wedge\cdots\wedge\mathfrak{O}^{(n-1)}$, $\mathfrak{O}^{(i)}$ the conjugates of $\mathfrak{O}$. When the condition holds $\mathfrak{O}^*$ is the totality of elements in K which are integral over $\mathfrak{o}$. An example is given of a ring $\mathfrak{O}$ in which the usual arithmetic holds but such that it does not hold in $\mathfrak{o}$. *N. Jacobson* (Chapel Hill, N. C.).

Nakayama, Tadasi. Normal basis of a quasi-field. Proc. Imp. Acad. Tokyo **16**, 532–536 (1940). [MF 4329]

As the reviewer has shown [Ann. of Math. (2) **41**, 1–7 (1940); cf. these Rev. **1**, 198], the fundamental theorem of the Galois theory is valid for a finite group $\mathfrak{G}$ of outer automorphisms acting in a quasi-field P. The present paper proves the existence of a normal basis (right or left) for P over Φ, the sub-quasi-field of invariant elements. This is equivalent to the theorem that the representation of the group $\mathfrak{G}$ in P is equivalent to the regular representation of $\mathfrak{G}$ over Φ. The proof is given in this second form and is an extension of Deuring's proofs [Math. Ann. **107**, 140–144 (1932)]. It does not require the semi-simplicity of the group ring and so is valid for any characteristic of P. The starting point of the argument is an extension of the well-known theorem of Hilbert-Speiser. *N. Jacobson.*

Lednew, N. Zu dem umgekehrten Problem der Galoisschen Theorie. Rec. Math. [Mat. Sbornik] N.S. **9** (**51**), 137–164 (1941). (Russian. German summary) [MF 4493]

The question of existence of normal extensions of finite algebraic number fields, with a preassigned Galois group G, is investigated by methods of algebraic geometry. The starting point is the Riemann existence theorem for algebraic functions of one variable with preassigned branch points and preassigned substitutions about the branch points. Let the substitutions generate a group simply isomorphic to G and let the branch points in the plane of the complex variable z be $\alpha_1, \alpha_2, \cdots, \alpha_r$. A field $\Omega(z, U)$ of algebraic functions (Ω is the field of complex numbers) is then uniquely determined, not, however, the primitive element U. A large part of the paper deals with an explicit construction of a family of functions U. Here use is made of the theorem of Riemann-Roch and of the polygon of Newton. The result of this construction is a defining equation $F(z, U)=0$ whose coefficients are polynomials, with integral coefficients, of certain parameters $x_1, x_2, \cdots, x_N$. Among these parameters occur the variable branch points $\alpha_1, \alpha_2, \cdots, \alpha_r$. The parameters x_i are not independent variables; they are connected by one algebraic relation $\Phi(x_1, x_2, \cdots, x_N)=0$,

with integral coefficients. [It is not clear to the reviewer why the condition that the polynomials $\psi_A(z)$ and $\psi_A'(z)$ have a common factor of degree not less than M is given by one equation (15), and not by a system of equations.] Thus to every point A of a sufficiently small region δ of the hypersurface $\Phi=0$ there corresponds a function $U=U_A$. Let k be a given finite field of algebraic numbers and let $k_A=k(x_1, x_2, \cdots, x_N)$. Moreover let $K_A(z)=k_A(z; U_A^{(1)}, \cdots, U_A^{(s)})$, where the $U_A^{(i)}$ are the roots of the equation $F(z, U)=0$. If k^* is the subfield of $K_A(z)$ consisting of the rational functions of z (with coefficients which are elements of the relative algebraic closure of k_A in $K_A(z)$), then the monodromy group G is the Galois group of the field $K_A(z)$ with respect to the field k^*. The region δ has been so chosen that for every point A in δ the equation $F(0, u)=0$ has only simple roots, and these roots have been included among the parameters x_i. Because of this it turns out that k_A is maximally algebraic in $K_A(z)$ and hence G is the Galois group of $K_A(z)$ with respect to $k_A(z)$. By Hilbert's irreducibility theorem there exist infinitely many specializations $z\to\omega$ (ω an integer) such that the Galois group of the resulting field $K_A(\omega)$ over k_A is still G. From this follows the main theorem of the paper: in order that there should exist a normal extension of k with G as Galois group, it is sufficient that the region δ contain at least one point A whose coördinates are in k. As a corollary of this result it is shown that, if k is the field of all algebraic numbers, then the field $k(z)$ has infinitely many normal extensions with a preassigned Galois group. *O. Zariski* (Baltimore, Md.).

Zariski, Oscar. Pencils on an algebraic variety and a new proof of a theorem of Bertini. Trans. Amer. Math. Soc. 50, 48–70 (1941). [MF 4869]

Let Σ be a field of algebraic functions of r variables over K as a field of constants. Suppose that V is a normal projective model of Σ with $\xi_1, \cdots, \xi_n$ as the coordinates of a nonhomogeneous general point. Then each $(r-1)$-dimensional subvariety $\Gamma\subset V$ determines uniquely a prime divisor of the first kind $\mathfrak{P}$, that is, a discrete rank one valuation B of Σ/K whose valuation ring is the quotient ring of V with respect to Γ. The arithmetic interpretation of a pencil $\{W\}$ of V_{r-1}'s on V reads as follows. A pencil is determined by a subfield P/K of Σ/K so that P/K has transcendence degree 1. Then a prime divisor $\mathfrak{P}$ induces in P/K either an isomorphism or a prime divisor $\mathfrak{p}$. The latter case is of interest for the construction of the pencil $\{W\}$ determined by P/K. A given $\mathfrak{p}$ of P/K arises from a finite number $\mathfrak{P}_1, \cdots, \mathfrak{P}_m$ of prime divisors of first kind. Letting t be a uniformizing variable of $\mathfrak{p}$ then the values $\mathfrak{P}_i(t)=h_i>0$ determine the multiplicities of the varieties Γ_i belonging to $\mathfrak{P}_i$ which are to correspond to $\mathfrak{p}$. The author sets $W_\mathfrak{p}=\sum h_i\Gamma_i$ and $\{W\}$ is the set of all $W_\mathfrak{p}$ for variable $\mathfrak{p}$. The explicit construction of $W_\mathfrak{p}$ for given $\mathfrak{p}$ is described later. A pencil is termed linear if the associated field is rational. Moreover, a pencil $\{W\}$ is composed with a pencil $\{W^*\}$ if $P\subset P^*$. Using valuation theory it is shown that each member of $\{W\}$ is the sum of a finite number of elements in $\{W^*\}$. The pencil $\{W\}$ is composite if and only if P is not algebraically closed in Σ. To obtain the classical definition of a pencil the prime divisors $\mathfrak{p}$ must be restricted to those of degree one over K. This observation is employed to define a general linear system $|W|$ of dimension s in Σ. Select s elements t_i in Σ such that $1, t_1, \cdots, t_s$ are linearly independent over K. Then $c_0+\sum c_it_i$, $c_0, c_i\in K$ can be written as a divisor $\mathfrak{A}_{(c)}\mathfrak{A}_\infty^{-1}$ with integral $\mathfrak{A}_{(c)}$, $\mathfrak{A}_\infty$ defined relative to the normal variety, $\mathfrak{A}_{(c)}$, $\mathfrak{A}_\infty$ being relatively prime in the sense of quasi-equality. Then the $\mathfrak{A}_{(c)}$ define for variable coefficients c_0, c_i the elements $W_{(c)}$ of an s-dimensional linear series $|W|$ on V. This system is free from fixed components, by construction of $\mathfrak{A}_{(c)}$ and $\mathfrak{A}_\infty$. The definition of a composite linear system of dimension s is similar to the one of composite pencils. The author proves the following theorem of Bertini: "A reducible linear system $|W|$, free from fixed components, is necessarily composite with a pencil. If K is maximally algebraic in Σ, then the assumption that $|W|$ is reducible can be replaced by the weaker assumption that $|W|$ is absolutely reducible." A necessary prerequisite is the theorem for pencils: "If a pencil free from fixed components is not composite, then all but a finite number of members of the pencil are irreducible varieties over K." The essential method of proof is a reduction to algebraic extensions of function fields of one variable and an application of the theory of specialization. For this purpose the author adjoins to Σ the transcendentals $\eta_i=\sum u_{ij}\xi_i$, u_{ij} independent trancendentals, and obtains a field Σ^*. The field Σ^* contains $P^*=P(\{u_{ij}\}, \{\eta_i\})$ and $K^*=K(\{u_{ij}\}, \{\eta_i\})$. By construction Σ^*/P^* is algebraic and P^* has degree of transcendency 1 over K^*. A general lemma on the prolongation of valuations to pure transcendental extensions implies that each $\mathfrak{p}$ on P/K has a unique prolongation $\mathfrak{p}^*$ on P^*/K^*. Then $\mathfrak{p}^*=\prod_{i=1}^{s}\mathfrak{P}_i^{*h_i}$ in Σ^*. The key for proof is embodied in the following assertions: (A) each $\mathfrak{P}_i^*$ induces in Σ a prime divisor $\mathfrak{P}_i$ of 1st kind; (B) the $\mathfrak{P}_i$ are pairwise distinct; (C) if Γ_i is the irreducible variety on V belonging to $\mathfrak{P}_i$ then $\sum h_i\Gamma_i$ is the member $W_\mathfrak{p}$ of the pencil $\{W\}$ which corresponds to the place $\mathfrak{p}$ of P/K. *O. F. G. Schilling.*

Weil, André. On the Riemann hypothesis in function-fields. Proc. Nat. Acad. Sci. U. S. A. 27, 345–347 (1941).

Let K/k be a field of algebraic functions of one variable over the algebraically closed field k. The field K has a projective non-singular model Γ since it has separable generations. The correspondences $C, D, \cdots$ on K may be represented by elements of the additive group of divisors $\mathfrak{C}$ on the product surface $\Gamma\times\Gamma$. The surface can be considered as the ordered pair (P, Q) of two generic points P, Q of Γ. To fixed points $A, B\in\Gamma$ there correspond by means of the varieties (P, A), (B, P) singular correspondences Γ_A, Γ_B' of $\mathfrak{C}$. A rational function ϕ on $\Gamma\times\Gamma$ determines uniquely with respect to $\Gamma\times\Gamma$ a divisor C_ϕ consisting of its zeros and poles. Since $\Gamma\times\Gamma$ is free from singularities the intersection number $[C, D]$ is defined for any two distinct irreducible elements C, D of $\mathfrak{C}$. Then the degrees $r(C)$, $s(C)$ and the coincidence number $f(C)$ of a correspondence C can be interpreted by means of intersection numbers. The divisors C_0 which are linear combinations of divisors C_ϕ, Γ_A, Γ_B' form a two-sided ideal $\mathfrak{C}_0$ of $\mathfrak{C}$, where multiplication in $\mathfrak{C}$ is defined by successive application of correspondences. The difference ring $\mathfrak{R}=\mathfrak{C}-\mathfrak{C}_0=\{\gamma, \delta, \cdots\}$ is the ring of complex multiplications in K. The unit 1 of $\mathfrak{R}$ is the map of correspondence $\Delta=(P, P)$. Since $(P, Q)\rightleftarrows(Q, P)$ is a birational transformation of $\Gamma\times\Gamma$ it follows that R has an anti-automorphism $\gamma\to\gamma'$. By a judicious selection of the representative C for a given element γ it can be achieved that the components of $r(C)+s(C)-f(C)=\sigma(\gamma)$ are defined. The integral-valued function σ has the following properties: (i) $\sigma(n\cdot 1)=2gn$, where g is the genus of K and n an integer; (ii) $\sigma(\gamma)=\sigma(\gamma')$; and (iii) $\sigma(\gamma\delta)=\sigma(\delta\gamma)$. All these properties are derived by means of the theory of intersection numbers. The author indicates next that $\sigma(\gamma\gamma')\geqq 0$, $\sigma(\gamma\gamma')=0\leftrightarrow\gamma=0$. The proof

of this fundamental inequality [see C. R. Acad. Sci. Paris 210, 592–594 (1940); these Rev. 2, 123] uses the notion of the generic complementary correspondence. The final proof of the Riemann hypothesis then follows readily.
O. F. G. Schilling (Chicago, Ill.).

Riblet, Henry J. Algebraic differential fields. Amer. J. Math. 63, 339–346 (1941). [MF 4154]
The following analogue of the symmetric function theorem is proved: A rational integral function in $y_1, \cdots, y_n$ and their derivatives up to order l that is invariant under all permutations of the symmetric group is expressible in one and only one way as a rational integral function of the elementary symmetric functions, their derivatives up to order l and a negative power of the discriminant of the y's. The author also derives some properties of algebraic extensions of differential fields. For example let $\mathfrak{K}$ be a field in which a differentiation process is defined and suppose that its constant field K is algebraically closed of characteristic 0. If $y^n+a_1y^{n-1}+\cdots+a_n=0$ is an irreducible equation over $\mathfrak{K}$ and $\theta_1, \cdots, \theta_n$ are its roots, then the pth derivatives $\theta_r^{(p)} \neq \theta_s^{(p)}$ if $r \neq s$. *N. Jacobson* (Chapel Hill, N. C.).

Riblet, Henry J. A differential basis for algebraic differential fields. Amer. J. Math. 63, 347–351 (1941). [MF 4155]
It is shown that, if $\mathfrak{K}$ is a field of characteristic 0 in which a differentiation process is defined, then in any algebraic extension $\mathfrak{K}(\theta)$ of degree n there exists an element α whose n conjugates are linearly independent over the constants of $\mathfrak{K}(\theta_1, \cdots, \theta_n)$, θ_i the conjugates of θ. The elements $\alpha, \alpha', \cdots, \alpha^{(n-1)}$ then form a basis for $\mathfrak{K}(\theta)$ over $\mathfrak{K}$.
N. Jacobson (Chapel Hill, N. C.).

Kolchin, Ellis Robert. On the exponents of differential ideals. Ann. of Math. (2) 42, 740–777 (1941). [MF 4971]
Let Σ be an ideal of differential polynomials and $\{\Sigma\}$ the perfect ideal generated by Σ. If positive integers q exist such that the qth power of $\{\Sigma\}$ is a part of Σ, the least such q is called the exponent of Σ. If no q exists, the exponent is defined as ∞. The paper deals especially with the exponent of the ideal generated by a form A which involves one unknown and is of the first order in that unknown. Multiplicities are assigned to the singular solutions of A. The exponent is infinity whenever A has a singular solution of multiplicity greater than unity. When the multiplicities are all unity, the exponent, if A satisfies a weak restriction, is either one or two. The exponent is unity, in the "general case," when there are no singular solutions. Other results deal with chains of ideals and with the exponent of an ideal relative to any ideal of which it is a part. An appendix presents an abstract theory of the resolvent of a prime ideal.
J. F. Ritt (New York, N. Y.).

NUMBER THEORY

Gupta, Hansraj. An important congruence. Proc. Indian Acad. Sci., Sect. A. 13, 85–86 (1941). [MF 4350]
The author gives another proof, using primitive roots modulo p^n, of the theorem: The sum of the rth powers of the numbers less than and prime to p^n is divisible by p^{n-1} or p^n according as $p-1$ does or does not divide r. The theorem is stated for r even, but obviously holds for odd r.
D. H. Lehmer (Berkeley, Calif.).

Aucoin, A. A. Solution of a quartic Diophantine equation. J. Math. Pures Appl. (9) 20, 17–21 (1941). [MF 4637]
A particular quartic equation in four integral unknowns is studied. Each side of this Diophantine equation is composed of two linear and one quadratic factor. All integral solutions proportional to a given solution are found.
I. A. Barnett (Cincinnati, Ohio).

Aucoin, A. A. Solution of a quartic Diophantine equation. Bol. Mat. 14, 36–39 (1941). [MF 4284]
The author considers the quartic Diophantine equation $\Pi_{i=1}^{4}\sum_{j=1}^{3}a_{ij}x_j=f(y_1, y_2, \cdots, y_n)$, where $f(y)$ is a given polynomial with integral coefficients of degree $n \not\equiv 0 \pmod 4$. He shows that all integral solutions, apart from those satisfying certain linear relations, are given (in a sense discussed in the paper) by three sets of formulas according as $n-1$, $n-2$, $n-3$ is divisible by 4, respectively. The solutions depend in a complicated manner upon the determinant and cofactors of the integral matrix $\|a_{ij}\|$ and upon $a+3$ integral parameters. *I. A. Barnett* (Cincinnati, Ohio).

Billing, G. A Diophantine equation with seven solutions. Ark. Mat. Astr. Fys. 27A, no. 14, 7 pp. (1941). [MF 4354]
La courbe $x_2^2x_3=x_1^3-43x_1x_2^2+166x_3^3$ possède exactement les sept points rationnels (0, 1, 0), (3, ±8, 1), (−5, ±16, 1), (11, ±32, 1). Pour la démonstration on applique avant tout un théorème établi par l'auteur dans sa thèse [Uppsala, 1938] qui permet, au moyen d'une factorisation dans le corps déterminé par la racine réelle de l'équation $\xi^3-43\xi+166=0$, d'assurer que le nombre des solutions rationnelles est fini et impair; un théorème de T. Nagell permet alors d'affirmer que toutes les solutions doivent correspondre à la valeur de x_3 0 ou 1, et à x_2^2 diviseur du discriminant de cette équation. La discussion des cas que par lá résultent encore possibles est omise. On signale une observation fondée sur la considération du paramètre elliptique qui permet de réducir notablement ces tentatives. *B. Levi* (Rosario).

Billing, G. A Diophantine equation with nine solutions. Ark. Mat. Astr. Fys. 27B, no. 8, 5 pp. (1941). [MF 4356]
Par la même méthode de la note précédente, l'auteur démontre que la courbe $x_2^2x_3=x_1^3-219x_1x_3^2+1654x_3^3$ possède exactement les 9 points rationnels (0, 1, 0), (3, ±32, 1), (11, ±24, 1), (−13, ±48, 1), (35, ±192, 1). *B. Levi.*

Ko, Chao. Note on the Diophantine equation $x^y y^x = z^z$. J. Chinese Math. Soc. 2, 205–207 (1940). [MF 4207]
The reviewer conjectured that the equation $x^y y^x = z^z$ has no solutions in integers, $x, y, z > 1$. The author proves that this conjecture holds only if $(x, y)=1$. If $(x, y) \neq 1$ there are infinitely many solutions, say

$$x=2^{2^{n+1}(2^n-n-1)+2n}(2^n-1)^{2(2^n-1)},$$
$$y=2^{2^{n+1}(2^n-n-1)}(2^n-1)^{2(2^n-1)+2},$$
$$z=2^{2^{n+1}(2^n-n-1)+n+1}(2^n-1)^{2(2^n-1)+1}.$$

P. Erdös (Philadelphia, Pa.).

Sastry, S. On Tarry's problem. J. Indian Math. Soc. (N.S.) **4**, 167–168 (1940). [MF 4321]

The problem of finding, for a fixed k, sets of integers x_i, y_i $(i=1, 2, \cdots, n)$ such that $\sum_{i=1}^{n} x_i^m = \sum_{i=1}^{n} y_i^m$, for all $m=1, 2, \cdots, k$, is known as Tarry's problem. It is only interesting to consider the case where all x_i are different from all y_i, and it is obvious that the least possible number n for a given k is then $k+1$. For $1 \leqq k \leqq 7$ solutions are known for this least value of n. The author constructs a new solution for $k=7$ with $n=8$, by a method similar to that used by Tarry [see Dickson, History of the Theory of Numbers, vol. 2, p. 710]. From this he deduces the relation $9^{2k}+47^{2k}+49^{2k}+67^{2k}=23^{2k}+31^{2k}+61^{2k}+63^{2k}$, for all $k=1, 2, 3$.

H. W. Brinkmann (Swarthmore, Pa.).

Trost, Ernst. Über eine Verallgemeinerung eines Satzes von Fermat. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 138–142 (1940). [MF 4411]

If $M=N(N+1)$, where N is a natural number, the Diophantine equation $(Mx^2-N)^2=My^2-N$ has the trivial solution $x=0, y=1$, and it is readily seen that $x=2, y=4N+1$ is also a solution. The author proves that, if $N=D^2$ $(D \neq 1)$, these are the only solutions in non-negative integers. For $D=1$ this is a well-known conjecture of Fermat, which was proved by Genocchi [Nouv. Ann. Math. (3) **2**, 306–310 (1883)]. The proof is accomplished by means of a theorem about the equation $Ax^4-By^4=\pm 1$ which Ljunggren proved by using the theory of units. *H. W. Brinkmann.*

Ribeiro, Hugo. On a problem of Tchebycheff. Univ. Lisboa. Revista Fac. Ci. **2**, 89–92 (1940). (Portuguese) [MF 4306]

The author finds that the probability that a integers chosen at random are relatively prime is $1/\zeta(a)$. The argument leading to the result is purely heuristic.

M. Kac (Ithaca, N. Y.).

Wintner, Aurel. On the distribution function of the remainder term of the prime number theorem. Amer. J. Math. **63**, 233–248 (1941). [MF 4146]

Let $\psi(x)=\sum_{p^m \leqq x} \log p$. Then a classical result of Littlewood states that for infinitely many x

$$\psi(x)-x > cx^{\frac{1}{2}} \log\log\log x$$

and

$$\psi(x)-x < -cx^{\frac{1}{2}} \log\log\log x.$$

The author proves the following theorem: Let us assume the Riemann hypothesis; then

$$h(t)=\psi(x)-x/x^{\frac{1}{2}}, \qquad x=e^t,$$

has a distribution function, which has an unbounded spectrum and moments of arbitrarily high order. The proof depends on the fact proved a few years ago by the author [Amer. J. Math. **57**, 534–538 (1935)] that the trigonometric series occurring in the explicit formula of Riemann is actually the Fourier expansion of the function which it represents. (The Fourier character is meant in the sense (B^2) of the theory of almost periodic functions.) *P. Erdös.*

Gut, Max. Mittel aus Dirichlet-Reihen mit reellen Restcharakteren. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 214–224 (1940). [MF 4418]

The author [Monatsh. Math. Phys. **48**, 153–160 (1939); these Rev. **1**, 136] has discussed $\lim_{i\to\infty} (\zeta_i(s))^{1/n_i}$, where $\zeta_i(s)$ is the Dedekind zeta-function in the field k_i, k_0 is the rational field, each k_i is formed by adjoining an absolute cyclic field of prime degree q to k_{i-1}, and n_i is the absolute degree of k_i. In this paper the results for $q=2$ are obtained without use of the theory of algebraic fields of infinite order, and further results are obtained. The numbers $d_1, d_2, \cdots$ are 1 and the discriminants of all quadratic fields, written in a certain partially determined order; $L(s, d)$ is the Dirichlet L-series with character (d/n). Taking $\Re(s)>1$ the author determines the limits, as i tends to infinity, of the geometric, arithmetic and harmonic means of $L(s, d_1), \cdots, L(s, d_i)$. The proofs consist mainly of a consideration of certain properties of the quadratic residue symbol (d/n).

H. S. Zuckerman (Seattle, Wash.).

Anfertieva, E. A. Sur les formules sommatoires et les identités analytiques, liées à une classe de fonctions arithmétiques. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 391–393 (1941). [MF 4377]

The author considers the Dirichlet series

$$\zeta(\rho-\nu)\zeta(\rho+\nu)+\sum_{n=1}^{\infty} a_\nu(n)/n^\rho,$$

where

$$a_\nu(n)=\sum_{d\delta=n} d^{2\nu}/n^\nu,$$

and obtains identities for this function. The detailed proofs will be given in a subsequent paper. *P. Erdös.*

Koschliakov, N. Application of Mellin's transformation to the deduction of some summation formulae. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 43–56 (1941). (Russian. English summary) [MF 4269]

Mellin's inversion formulae

$$F(s)=\int_0^\infty f(x)x^{s-1}dx, \quad f(x)=\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \frac{F(s)ds}{x^s}, \quad x>0,$$

with $F(s)=F(\tau+it)$ suitably restricted, and the calculus of residues are used to give new proofs of three summation formulae (associated with Plana-Abel-Cauchy, Fourier and Voronoi):

$$\tfrac{1}{2}f(0)+\sum_1^\infty f(\nu)=\int_0^\infty f(x)dx-2\int_0^\infty \frac{f(ix)-f(-ix)}{2i}\frac{dx}{e^{2\pi x}-1},$$

$$f(x)=\int_0^1 f(\tau)d\tau+2\sum_{\nu=1}^\infty \int_0^1 f(\tau)\cos 2\nu\pi(\tau-x)d\tau,$$

$$\sum_{\alpha<n\leqq\beta}^{n} d(n)f(n)$$

$$=\tfrac{1}{2}d(\beta)f(\beta)-\tfrac{1}{2}d(\alpha)f(\alpha)+\int_\alpha^\beta [2C+\log x]f(x)dx$$

$$+2\pi\sum_{n=1}^\infty d(n)\int_\alpha^\beta f(x)\{-Y_0(4\pi(nx)^{\frac{1}{2}})+2\pi^{-1}K_0(4\pi(nx)^{\frac{1}{2}})\}dx,$$

where $d(n)$ denotes the number of divisors of n.

G. Pall (Montreal, Que.).

Hua, Loo-keng. On an exponential sum. J. Chinese Math. Soc. **2**, 301–312 (1940). [MF 4215]

Let $f(x)=a_kx^k+\cdots+a_1x$, $(a_1, \cdots, a_k, q)=1$,

$$S_q=\sum_{x=1}^{q} \exp(2\pi i f(x)/q).$$

Then $|S_q| < C_{k,\epsilon}q^{1+\epsilon-1/k}$. The proof amplifies slightly and

corrects (notably in the case ind $f(x)=$ind $f(px)$) that in C. R. Acad. Sci. Paris **210**, 520–523 (1940) [these Rev. **2**, 40]. However he should on p. 303 make the subdivision $l\leqq$ and $l>2t+1$, and not $t+1$. He proves next

$$\sum_{x=1}^{m}\exp(2\pi i f(x)/q)=(m/q)S_q+O(q^{1+\epsilon-1/k}).$$

Also, let $f(x)$ be any integral-valued polynomial, d be the least common denominator of the coefficients of f, $q=(q,d)$, $\alpha=(a/q)+\beta$, $q=O(P^{1-\epsilon})$, $|\beta|=O(q^{-1}P^{-k+1-\epsilon})$; then

$$\sum_{x=0}^{P}e^{2\pi i\alpha f(x)}=(1/q)\Big(\sum_{x=1}^{q}e^{2\pi i\alpha f(x)/q}\Big)\int_0^P e^{2\pi i f(x)\beta}dx+O(q^{1+\epsilon-1/k}).$$

G. Pall (Montreal, Que.).

Hua, Loo-keng. On a generalized Waring problem. II. J. Chinese Math. Soc. **2**, 175–191 (1940). [MF 4204]

In part I [Proc. London Math. Soc. (2) **43**, 161–182 (1937)] and elsewhere [J. Chinese Math. Soc. **1**, 21–61 (1936); Quart. J. Math., Oxford Ser. **9**, 199–202 (1938)], Hua obtained a formula

$$a^{-s/k}(\Gamma(1+1/k)/\Gamma(s/k))\mathfrak{S}(N)N^{s/k-1}+O(N^{s/k-1-\rho})$$

for the number of representations of N as $P(x_1)+\cdots+P(x_s)$ in integers $x_i\geqq 0$. Here $P(x)=ax^k+\cdots$ is an integral-valued polynomial of degree $k(\geqq 3)$, and $\rho(>0)$ is independent of N; and $s\geqq 10k^3\log k$ if $k>15$, $s\geqq 2^k+1$ if $3\leqq k\leqq 15$. He now proves that the singular series $\mathfrak{S}(N)$ exceeds a positive constant independent of N if $s\geqq(k-1)2^{k+1}$, whence $G(P)\leqq(k-1)2^{k+1}$, a considerable improvement on his result in I. For the special polynomial $H(x)=2^{k-1}F_k(x)-2^{k-2}F_{k-1}(x)+\cdots+(-)^{k-1}F_1(x)$, where $F_i(x)=x(x-1)\cdots(x-i+1)/i!$, he proves if $k\geqq 5$ that precisely $G(H)=2^k-\frac{1}{2}\{1-(-1)^k\}$.

G. Pall (Montreal, Que.).

Hua, Loo-Keng. On Waring's problem with cubic polynomial summands. J. Indian Math. Soc. (N.S.) **4**, 127–135 (1940). [MF 4317]

It is proved [cf. C. R. Acad. Sci. Paris **210**, 650–652 (1940); these Rev. **2**, 35] that every large N is a sum of eight values of $P(x)=a(x^3-x)/6+\frac{1}{2}b(x^2-x)+cx+d$ for integers $x\geqq 0$. Here a, b, c are integers, $(a,b,c)=1$, $a>0$.

G. Pall (Montreal, Que.).

Olds, C. D. On the representations, $N_3(n^2)$. Bull. Amer. Math. Soc. **47**, 499–503 (1941). [MF 4545]

Let any integer $n=2^kPQ$, where the prime factors (if any) of Q are of the form $4x+3$, while those of P are of the form $4x+1$. Let the actual factorization of Q be

$$Q=\prod_{\nu=1}^{R}q_\nu^{b_\nu}.$$

Then the number $N_3(n^2)$ of representations of n^2 as the sum of three squares (with the usual conventions as to their enumeration) is given by the formula

$$N_3(n^2)=6P\prod_{\nu=1}^{R}\left\{q_\nu^{b_\nu}+2\frac{q_\nu^{b_\nu}-1}{q_\nu-1}\right\}.$$

This formula is credited to Stieltjes, who used elliptic functions. A proof of it using integral quaternions has been given recently by Pall [Trans. Amer. Math. Soc. **47**, 487–500 (1940); cf. these Rev. **2**, 36]. A completely elementary arithmetical proof, based on the methods of Liouville, is given in the present paper. *D. H. Lehmer.*

Buchstab, A. A. Sur la décomposition des nombres pairs en somme de deux composantes dont chacune est formée d'un nombre borné de facteurs premiers. C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 544–548 (1940). [MF 3968]

Let $P_\omega(x;x^{1/\alpha})$ denote the number of non-negative integers not exceeding x which do not belong to any of the progressions a_0+kp_0, a_i+kp_i, b_i+kp_i, $i=1,2,\cdots$, where $p_0=2$, $3\leqq p_i<x^{1/\alpha}$; a_i and b_i are fixed integers such that $0\leqq a_i<p_i$, $0\leqq b_i<p_i$, $a_i\neq b_i$, and ω is the domain of values of a_i and b_i. In the Brun sieve method inequalities such as

$$P_\omega(x;x^{1/\alpha})>\lambda(\alpha)cx/\log^2 x+O(x/\log^3 x)$$

are derived and used to show, for example, that all large even integers are sums of two integers each having not more than a certain number of prime factors. The smaller α can be chosen the better the result. In this paper the author shows how to construct other λ-functions with smaller α from known ones. The result obtained is that all large even integers are sums of two integers having at most four prime factors. *R. D. James* (Saskatoon, Sask.).

Ko, Chao. On the Meyer's theorem and the decomposition of quadratic forms. J. Chinese Math. Soc. **2**, 209–224 (1940). [MF 4208]

It is proved first that an n-ary form $f=\sum a_{ij}x_ix_j$ (a_{ij} integers) of determinant ± 1 has minimum not greater than 1, if $n\leqq 7$, and represents zero if its signature is 1. A proof of Meyer's theorem that indefinite forms in five or more variables represent zero is then obtained by imbedding a related form as a section of a form of determinant ± 1 in 7 variables. Similarly by imbedding an n-ary form in a form of determinant 1 in $n+3$ variables it is proved that every n-ary integral f can be expressed as an algebraic sum of $n+3$ squares of linear forms with integer coefficients. The proof uses the author's work [Acta Arith. **3**, 79–85 (1938)] on quadratic forms of determinant 1, and a theorem by Zilinskas [J. London Math. Soc. **13**, 225–240 (1938)] that every properly primitive indefinite quadratic form of determinant ± 1 is equivalent to a form $\sum x_i^2-\sum x_j^2$.

G. Pall (Montreal, Que.).

Linnik, U. Über die Darstellung grosser Zahlen durch positive ternäre quadratische Formen. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 363–402 (1940). (Russian. German summary) [MF 4080]

[In the title of the German summary the word "grosser" was incorrectly translated as "ganzer." For the author's previous work, cf. these Rev. **2**, 36.] The fundamental result concerns quaternions, an extension to generalized quaternions being sketched in the second part. Let p be an odd prime, m large, $m\neq 4k$ or $8k+7$, $(-m|p)=1$, $x_0^2\equiv -m \pmod p$; as x ranges over all proper, pure quaternions $i_1x_1+i_2x_2+i_3x_3$ of norm m, every quaternion of norm p occurs as a divisor of x_0+x. The proof is ingenious but has a serious error, which the reviewer has corrected only with considerable effort: Linnik uses the false result that the number of representations as a sum of three squares of a binary quadratic form of determinant D is $O(D^\epsilon)$. The connection between classes of binary quadratic forms of determinant m and pairs (x,y) of vectors of norm m is used to count the number of such pairs in which x_0+x and x_0+y have conjugate divisors of norm p^s, where s is variable with m: $m^{1+\tau}\leqq p^s<pm^{1+\tau}$, $\tau=-\frac{1}{2}\log(1-p^{-1})/\log p$. Net result: the number is $O(m^{1+\epsilon})$. But the assumption that any divisor of norm p fails to occur makes the number of larger

order than $m^{1+\epsilon}r^{\epsilon-1}$. Application: if f is a positive primitive ternary of invariants $[p, 1]$, such that $(f|p)=(-1|p)$ [whence, it might be remarked, f is derived from a sum of three squares by a transformation of determinant p], then (our above) m is represented by f. The extension involves any form of invariants $(\Omega, 1)$, where Ω is odd, and, for at least one prime p dividing Ω, $(f|p)=(-1|p)$; then every large m satisfying the generic conditions and prime to Ω is represented. The same error is made in the extension, but perhaps can be adjusted. *G. Pall* (Montreal, Que.).

Linnik, U. V. "The large sieve." C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 292–294 (1941). [MF 4382]

The Viggo Brun sieve method enables one to find bounds for the number of integers not greater than X which do not belong to k given residue classes modulo p_i, where the p_i are the primes not exceeding $\sqrt{X}$. The method does not apply if the fixed number k is replaced by $f(p_i)$, where $f(p_i)$ increases with p_i. A method of dealing with this case is discussed in the present paper. It involves the exponential sum

$$S(\alpha)=\sum_{j=1}^{Z} e^{2\pi i a M_j}$$

and an estimation of the integral

$$\int_0^1 |S(\alpha)|^2 d\alpha = Z.$$

R. D. James (Saskatoon, Sask.).

Scherk, Peter. Two estimates connected with the (α, β)-hypothesis. Ann. of Math. (2) **42**, 538–546 (1941). [MF 4300]

Let A, B, C be sets of positive integers a, b, c, respectively, and $A(x)$, $B(x)$, $C(x)$ their respective number functions. Let C be the set of all numbers of the form a, b, $a+b$. The so-called $\alpha+\beta$ hypothesis states: Let α and β be two positive numbers with $\alpha+\beta\leqq 1$. Let $A(x)\geqq\alpha x$, $Bx\geqq\beta x$, $x=1, 2, \cdots, n$. Then $C(x)\geqq(\alpha+\beta)x$. The author obtains two new estimates for $C(x)$ in terms of $A(x)$ and $B(x)$.

P. Erdős (Philadelphia, Pa.).

Rosser, Barkley. A generalization of the Euclidean algorithm to several dimensions. Proc. Nat. Acad. Sci. U. S. A. **27**, 309–311 (1941).

A set of vectors $U_1, U_2, \cdots, U_r$ is said to be a G.C.F. of a given set of vectors $V_1, V_2, \cdots, V_s$ if the U's are linearly independent, each U is a linear combination of the V's and each V a linear combination of the U's, the combining coefficients in each case required to be integers. If the integral linear combination $\sum a_i V_i$ has no limit point (commensurable case), then it is stated that the V's always have a G.C.F. A procedure is outlined (generalization of the Euclidean algorithm for two vectors) for finding the G.C.F. of any set of commensurable vectors. For the case of two incommensurable vectors the process leads to the continued fraction algorithm. Mention is made of applications to problems in approximation, Diophantine equations and quadratic forms. *I. A. Barnett* (Cincinnati, Ohio).

***Delone, B. N. and Faddeev, D. K. Theory of Irrationalities of Third Degree.** Acad. Sci. URSS. Trav. Inst. Math. Stekloff, v. **11**, 1940. 340 pp. (Russian)

The purpose of this outstanding monograph is to present all that is known at the present time about cubic irrationalities and such problems in number theory as are intimately connected with them. The book for the most part consists of the original investigations of its authors, and even that which has been contributed by other investigators is presented from a new and original point of view. Two features are very characteristic of the mode of presentation: on the one hand the extensive use of geometrical considerations as a background for the true understanding of complicated situations which otherwise would remain obscure, and on the other hand, the care shown by the authors in inventing effective methods of solution, illustrated by actual application to numerical examples and to the construction of valuable tables.

The following survey of the book by chapters will convey only an imperfect idea of its very rich contents. Introductory Chapter I is devoted to an original geometric presentation of the general theory of algebraic numbers. It is based on considerations of multiplicative lattices, that is, lattices which contain the product of any two of their points, meaning by the product of two points a point whose coordinates result from multiplication of the corresponding coordinates of the given points. It is shown that properties of algebraic fields are reflected in properties of certain multiplicative lattices. Of special interest from this point of view is a new geometric interpretation of the Galois theory.

Chapter II deals with cubic fields. Besides such questions as determination of minimal bases and factorization of primes into ideals, it also contains things less generally known. Such is, for instance, the correspondence between binary cubic forms and rings in cubic fields, effective solution of the problem whether two given cubic equations determine the same field or not (inverse problem of Tschirnhausen), and as an application of this solution, a method to decide the equivalence of two proposed binary cubics without the theory of reduction.

Chapter III deals with tabularization and classification of cubic and biquadratic fields. In the first place a method based on geometric considerations is developed for tabulating all cubic rings whose discriminant does not exceed in absolute value a prescribed limit. Two tables are constructed according to this method: one giving all cubic rings and corresponding forms of positive discriminants not exceeding 1296, and another for negative discriminants not exceeding 1000 in absolute value. It is shown that cyclic rings can be tabulated by a much simpler method. Similar methods are applied to tabulation of biquadratic fields and the results are exhibited in three tables. In the same chapter one finds a geometric interpretation of binary cubics and their covariants and the theory of reduction based on this interpretation. The chapter ends with a very curious method for constructing cubic fields from their "supporting" quadratic fields, and, similarly, biquadratic fields from their "supporting" cubic fields.

Chapter IV brings a very beautiful geometric interpretation of Voronoi's algorithm for finding fundamental units in cubic rings of positive and negative discriminants as well as for deciding the equivalence of ideals. Two tables are appended to this chapter: one giving fundamental units in cubic fields of a negative discriminant not exceeding 379 in absolute value and another exhibiting fundamental units in fields $R(\sqrt[3]{a})$ for $a\leqq 70$.

Chapter V deals with the theorem of Thue as applied to cubic irrationalities and contains a simplified proof and some improvement of Siegel's result concerning the number of representations of integers by binary cubics with a positive discriminant. It is proved that when $f(x, y)$ is such a

form the inequality $|f(x, y)| \leqq k$ has at most 15 solutions in integers x, y (not distinguishing the two solutions x, y and $-x, -y$) if the discriminant is large compared to k.

The final chapter VI is perhaps the most interesting of the whole book. It is devoted to a problem, not yet completely solved, of the representation of integers by binary cubic forms. Since next to nothing is known in this respect for positive discriminants, only forms of negative discriminants are considered. After having shown that some particular equations such as $x^3+ay^3=1$, $ax^3+by^3=1$ can be completely solved, the following remarkable theorem is established: the number of representations of 1 by an irreducible binary cubic with negative discriminant never exceeds 5; there are 5 representations only by forms of discriminant -23 and 4 only by forms of discriminant -31 and -44. In the proof of this theorem use is made of a peculiar process called "algorithm of elevation" which in all numerical examples that the authors have considered always leads to the complete solution of the problem; but it remains an open question whether this will always be so. By this method representations of 1 by all non-equivalent binary cubics with discriminants not exceeding 300 in absolute value have been determined and are given in a particular table. The chapter and the whole book end with the proof of Mordell's theorem concerning rational points on cubics and a detailed discussion of the particular equation $x^3+y^3=Az^3$. For all values $A \leqq 50$ such solvable equations, with their fundamental solutions, are exhibited in a table.

J. V. Uspensky (Stanford University, Calif.).

Gelfond, A. On the simultaneous approximations of algebraic numbers by rational fractions. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 99–104 (1941). (Russian. English summary) [MF 4509]

The author establishes an exact connection between the simultaneous approximation of powers of an algebraic number by fractions with equal denominators and the structure of linear forms of powers of the number, provided that the form is sufficiently rapidly convergent to zero. Relations between linear forms involving arbitrary numbers and simultaneous approximation of these numbers were investigated by Khintchine (Khintchinesches Übertragungsprinzip) [Rend. Circ. Mat. Palermo **50**, 170–195 (1926)], but the relation for a certain class of forms is here determined in a very precise way.

D. C. Spencer.

Robinson, Raphael M. On the simultaneous approximation of two real numbers. Bull. Amer. Math. Soc. **47**, 512–513 (1941). [MF 4549]

Using Dirichlet's "pigeonhole principle" the author proves that for every pair of real numbers ξ_1 and ξ_2 and for every positive integer s there exist three integers a_1, a_2 and b such that $0<b \leqq s$ and

$$|b\xi_k - a_k| \leqq \max\left(\frac{[s^{\frac{1}{2}}]}{s+1}, \frac{1}{[s^{\frac{1}{2}}]+1}\right), \qquad k=1, 2.$$

This pair of simultaneous approximations is the best possible in the sense that for every s there exist values of ξ_1 and ξ_2 such that one of these inequalities would not be satisfied if the sign of equality were removed. The corresponding problem for the simultaneous approximation of more than two real numbers "appears more difficult."

D. H. Lehmer (Berkeley, Calif.).

Mahler, Kurt. An analogue to Minkowski's geometry of numbers in a field of series. Ann. of Math. (2) **42**, 488–522 (1941). [MF 4298]

Notation: $\mathfrak{P}$ an arbitrary field, z an indeterminate, $\mathfrak{T}$ the ring of all polynomials in z with coefficients in $\mathfrak{P}$, $\mathfrak{R}$ the field of all formal Laurent series $x=a_f z^f+a_{f-1}z^{f-1}+\cdots$ with coefficients $a_f, a_{f-1}, \cdots$ in $\mathfrak{P}$, P_n the n-dimensional space of all points $X=(x_1, \cdots, x_n)$ with coordinates $x_1, \cdots, x_n$ in $\mathfrak{R}$, Λ_n the set of all lattice-points in P_n (that is, that of all points with coordinates in $\mathfrak{T}$). The non-Archimedean valuation $|x|$ is defined as $|0|=0$ and $|x|=e^f$ if $a_f \neq 0$. A distance function is any function satisfying $F(0)=0$, $F(X)>0$ if $X \neq 0$, $F(tX)=|t|F(X)$ for all t in $\mathfrak{R}$, $F(X-Y) \leqq \max(F(X), F(Y))$; it is further assumed that $F(X)$ is an integral power of e, for all $X \neq 0$ in P_n. A convex body $C(\tau)$ is defined by the inequality $F(X) \leqq \tau$, for any $\tau>0$. Similar to the arithmetical definition of the volume of a body in the ordinary real space, a certain positive constant V may be defined as the volume of $C(1)$.

The following theorem is proved: There are n independent points $X^{(1)}, \cdots, X^{(n)}$ in Λ_n with the following properties: (1) $F(X^{(1)})$ is the minimum of $F(X)$ in all lattice points $X \neq 0$, and, for $k \geqq 2$, $F(X^{(k)})$ is the minimum of $F(X)$ in all lattice points X which are independent of $X^{(1)}, \cdots, X^{(k-1)}$. (2) The determinant of the points $X^{(1)}, \cdots, X^{(n)}$ is 1. (3) The numbers $F(X^{(k)})=\sigma^{(k)}$ satisfy the formulae $0<\sigma^{(1)} \leqq \sigma^{(2)} \leqq \cdots \leqq \sigma^{(n)}$, $\sigma^{(1)}\sigma^{(2)} \cdots \sigma^{(n)}=V^{-1}$. The result is in analogy to a well-known theorem of Minkowski [Geometrie der Zahlen, Leipzig, 1910, §§ 50–53], and the proofs in the paper are also based on the methods of Minkowski. Some special Diophantine problems in P_n are considered as examples.

C. L. Siegel (Princeton, N. J.).

Tschebotarëw, N. Beweis des Minkowski'schen Satzes über lineare inhomogene Formen. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 27–30 (1940). [MF 4400]

Let $y_k=b_k+\sum_{i=1}^{n}a_{ki}x_i$ $(k=1, \cdots, n)$ be n nonhomogeneous linear functions with real coefficients, where the determinant $D=|a_{ki}| \neq 0$. An unproved hypothesis due to Minkowski asserts that the inequality $|y_1y_2 \cdots y_n| \leqq 2^{-n}|D|$ has an integral solution $x_1, \cdots, x_n$. The author considers the following simpler problem: To determine a number K_n depending only on n and not on the coefficients b_k and a_{ki}, such that the inequality $|y_1y_2 \cdots y_n| \leqq K_n|D|$ has an integral solution. He proves that the lower bound k_n of admissible numbers K_n is not greater than $2^{-n/2}$. The paper was originally published in Russian [N. G. Tschebotareff, Sci. Notes Kazan Univ. **94**, 14–16 (1934)].

C. L. Siegel.

Mordell, L. J. Tschebotareff's theorem on the product of non-homogeneous linear forms. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 47–50 (1940). [MF 4403]

Proof of the inequality $k_n \leqq 2^{-n/2}\{1+(\sqrt{2}-1)^n\}^{-1}$ which is better than Tschebotareff's result $k_n \leqq 2^{-n/2}$ [cf. the preceding review].

C. L. Siegel (Princeton, N. J.).

Wintner, Aurel. On the lattice problem of Gauss. Amer. J. Math. **63**, 619–627 (1941). [MF 4687]

Let $r(n)$ be the number of solutions of the equation $x^2+y^2=n$ and let furthermore $R(x)=\sum_{n \leqq x}r(n)$ $(x \geqq 1)$ and $\bar{R}(x)=\frac{1}{2}\{R(x-0)+R(x+0)\}$. It is proved that the function $Q(t)=(\bar{R}(t^2)-\pi t^2)/t^{\frac{1}{2}}$ is almost periodic (B^2) for $\infty > t \geqq 1$, having $\pi^{-1}\sum n^{-\frac{1}{4}}r(n) \sin(2\pi n^{\frac{1}{2}}t-\pi/4)$ as its Fourier series.

Parseval's relation $M\{Q^2(t)\}=(2\pi)^{-1}\sum r(n)^2/n^1$ follows at once. The paper ends with some remarks on Waring's problem. *M. Kac* (Ithaca, N. Y.).

Bullig, G. Anwendung eines Iterationsverfahrens für diskrete Punktmengen auf Gitter (Z 2). Mitt. Math. Ges. Hamburg **8**, part 2, 164–187 (1940). [MF 4120]

This paper is the second of a series by the author on the geometry of numbers. In the first, designated Z 1, the author defines an iteration process applicable to sets of points M satisfying four axioms [Abh. Math. Sem. Hansischen Univ. **13**, 321–343 (1940); these Rev. **2**, 253]. Here the process is applied to a three-dimensional set G with the following property. The vector products of any two vectors of G are drawn from the origin. The ends of these vector products are a three-dimensional set G^*, and the origin is the only element of G and G^* which lies on the coördinate planes $x_1=0$, $x_2=0$, $x_3=0$. The totality M of the elements of G which lie in the octant $x_n>0$ $(n=1, 2, 3)$ satisfies the author's four axioms. The process (Ω_{pq}) is formulated as a linear (unimodular) transformation, and is illustrated with numerical examples. *D. C. Spencer* (Cambridge, Mass.).

ANALYSIS

v. Sz. Nagy, Bela. Über Integralungleichungen zwischen einer Funktion und ihrer Ableitung. Acta Univ. Szeged. Sect. Sci. Math. **10**, 64–74 (1941). [MF 4433]

Let $y(x)$ be absolutely continuous on $(-\infty, \infty)$ and let the integrals

$$J_\alpha=\int_{-\infty}^{\infty}|y|^\alpha dx, \quad K_p=\int_{-\infty}^{\infty}|y'|^p dx$$

exist for a given $\alpha>0$ and given $p\geqq 1$. Then

$$(J_{\alpha+\beta}/J_\alpha)^{1/\beta}\leqq[q/2H(q/\beta, (p-1)/p)]^{1/q}J_\alpha^{(p-1)/pq}K_p^{1/pq},$$

where $\beta>0$, $q=1+\alpha(p-1)/p$,

$$H(u, v)=\frac{(u+v)^{-(u+v)}\Gamma(1+u+v)}{u^{-u}\Gamma(1+u)v^{-v}\Gamma(1+v)}, \quad H(u, 0)=H(0, v)=1.$$

This inequality holds also in the limiting case $\beta=\infty$, where the left-hand member should be replaced by $\max_{-\infty<x<\infty}|y|$. Analogous inequalities, in case of a finite interval, have been recently discussed by E. Schmidt [Math. Ann. **117**, 301–326 (1940); cf. these Rev. **2**, 218]. In the present discussion the author partly uses methods of E. Schmidt.

J. D. Tamarkin (Providence, R. I.).

Boas, R. P., Jr. Functions with positive derivatives. Duke Math. J. **8**, 163–172 (1941). [MF 3951]

Let $n_1, n_2, \cdots$ be a given increasing infinite sequence of positive integers. Let $f(x)$ be a real function of class C^∞ in $-1<x<+1$, such that, for each $p=1, 2, \cdots$, $f^{(n_p)}(x)$ does not change sign in $(-1, 1)$. The author's first result is that, if n_{p+1}/n_p is bounded, then $f(x)$ is necessarily analytic in the range $-1<x<1$. The principal result of the paper is the construction of an example showing that this is about the best theorem in this direction. Indeed, let $n_{p+1}/n_p\to\infty$. Choose a fixed ϵ $(0<\epsilon<\frac{1}{2})$ and define a sequence λ_p by $\lambda_1=1$, $\lambda_{p+1}/\lambda_p=(n_p/n_{p+1})^{2n_p}\cdot\epsilon^{n_p}$. Finally let

$$f(x)=L+\tfrac{1}{2}\sum_{p=1}^{\infty}\lambda_p(T_{n_p}(x)+T_{n_p-1}(x)), \quad -1\leqq x\leqq 1,$$

where $T(x)=\cos(n\arccos x)$. It is shown that, if L is sufficiently large positive and ϵ sufficiently small, then $f(x)$ satisfies the following conditions: (1) $f(x)>0$ in $-1<x<1$ and is of class C^∞, (2) $f^{(n_p)}(x)>0$ $(-1<x<1; p=1, 2, \cdots)$, (3) $f(x)$ is not analytic at $x=0$. In fact the Taylor series of $f(x)$ at the origin is shown to have vanishing radius of convergence. [Nevertheless, the inequalities (2) imply that

$$\lim_{p\to\infty}\sum_{\nu=0}^{n_p-2}x^\nu(\nu!)^{-1}f^{(\nu)}(0)=f(x)$$

uniformly inside the open range $-1/(2e)<x<1/(2e)$, as readily shown by means of the author's Lemma 2 and Taylor's formula with integral remainder. The Taylor series of $f(x)$ is therefore an example of over-convergence (in the sense of Porter) of a divergent power series in the real domain.] *I. J. Schoenberg* (Philadelphia, Pa.).

Boas, R. P., Jr. and Pólya, G. Generalizations of completely convex functions. Proc. Nat. Acad. Sci. U. S. A. **27**, 323–325 (1941).

Following up a note by Pólya [the same Proc. **27**, 216–218 (1940); these Rev. **2**, 219] the authors clear up to a large extent problems raised by some work of S. Bernstein on regularly monotone functions and of Widder on completely convex functions [the same Proc. **26**, 657–659; these Rev. **2**, 219]. The main result quoted from the paper is as follows: "Theorem 1. Let $\{n_k\}$ and $\{q_k\}$ be sequences of positive integers, $\{n_k\}$ strictly increasing. Let $f(x)$ be real-valued and of class C^∞ in $-1\leqq x\leqq 1$. For $k=1, 2, \cdots$, let $f^{(n_k)}(x)$ and $f^{(n_k+2q_k)}(x)$ not change sign in $(-1, 1)$, and let $f^{(n_k)}(x)f^{(n_k+2q_k)}(x)\leqq 0$. (I) If $n_k-n_{k-1}=O(1)$ and $q_k=O(1)$, then $f(x)$ coincides in $(-1, 1)$ with an entire function of growth not exceeding order one and finite type. (II) If $n_k-n_{k-1}=O(n_k^\delta)$, $q_k=O(n_k^\delta)$, and $q_1+q_2+\cdots+q_k=O(n_k)$, where δ is fixed, $0<\delta<1$, then $f(x)$ coincides in $(-1, 1)$ with an entire function of finite order not exceeding $1/(1-\delta)$. (III) If $n_k-n_{k-1}=o(n_k)$, $q_k=o(n_k)$, and $q_1+q_2+\cdots+q_k=O(n_k)$, then $f(x)$ coincides in $(-1, 1)$ with an entire function." The special case $q_k=1$ generalizes Bernstein's results. Assuming $2q_k=n_{k+1}-n_k$, a wide generalization of Widder's theorem is obtained (Theorem 2). A third theorem states that these results are in certain sense the best possible.

I. J. Schoenberg (Philadelphia, Pa.).

Wall, H. S. Some recent developments in the theory of continued fractions. Bull. Amer. Math. Soc. **47**, 405–423 (1941). [MF 4531]

This paper is the printed version of an address delivered before the Detroit meeting of the American Mathematical Society, November, 1940. The author discusses in some detail some of the more recent results of his and his collaborators' research. This paper will be of interest to all workers in the field, and it is only to be regretted that it does not include a more extensive bibliography.

W. Leighton (Houston, Tex.).

Theory of Sets, Theory of Functions of Real Variables

Rothberger, Fritz. Sur les familles indénombrables de suites de nombres naturels et les problèmes concernant la propriété *C*. Proc. Cambridge Philos. Soc. **37**, 109–126 (1941). [MF 4238]

For two sequences $\sigma'=\{x_n'\}$, $\sigma''=\{x_n''\}$ of positive integers, the sequences considered being infinite and consisting exclusively of positive integers, the relation $\sigma' \dashv \sigma''$ means that $x_n' \leqq x_n''$ for every n. The sequence $\{u_n\}$ is a majorant for the family Φ of sequences if $\{x_n\} \in \Phi$ implies $\{x_n\} \dashv \{u_n\}$. A family of sequences without majorant is unbounded. The paper continues the study [Fund. Math. **32**, 294–300 (1939)] of implications of the following proposition $B_{(\aleph_\xi)}$: Every family of sequences of positive integers of cardinal $\aleph_\xi$ is bounded. Various equivalences are deduced likening those in Sierpiński's "Hypothèse du Continu," and, among others, results are obtained relating to problems of Hausdorff and of Sierpiński. Some of the results: (a) The following propositions are equivalent to one another: $B_{(\aleph_\xi)}$; (B_1) Every set of cardinal $\aleph_\xi$ composed of irrational numbers has property Θ, that is, is contained in an F_σ composed exclusively of irrational numbers. (B_3) The sum of $\aleph_\xi$ sets of property Θ is of property Θ. (B_6) Every real function of Baire is at most of class 1 on every set of cardinal $\aleph_\xi$. (b) $B_{(\aleph_1)}$ is equivalent to each of the following propositions: (1) There exists a family of sequences of positive integers which is unbounded and normally ordered as an Ω order according to the relation $\dashv$. (2) There exists a non-denumerable, concentrated set (E is concentrated if there exists a denumerable set D such that every G_δ containing D contains all of E with the exception of an at most denumerable set). (c) The property of being concentrated is not topologically invariant if there exist non-denumerable, concentrated sets. (d) (relating to a problem of Hausdorff) Proposition non-$B_{(\aleph_1)}$ is equivalent to the existence of a gap of type (Ω, ω^*) in the family of all the sequences of positive integers.

H. Blumberg (Columbus, Ohio).

Inagaki, Takeshi. Quelques propriétés des espaces abstraits. J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **9**, 193–208 (1940). [MF 4145]

The author continues his investigation [same J. **8**, 25–46, 145–162 (1939–1940); cf. these Rev. **1**, 301] of the relations between cardinal number and topology in Hausdorff space. Let H denote such a space and H_1 a space satisfying the first denumerability axiom. Let $M(\aleph_\alpha)$ be the set of $p \in M$ such that, for each neighborhood $U(p)$, $U(p)M$ has power not less than $\aleph_\alpha$. A set E is called hypercondensed in itself if each non-denumerable $M \subset E$ is condensed in itself. The following are some results. In H, if isolated sets are at most denumerable, then $MM(\aleph_1)=0$ implies that the power of $M \leqq \aleph_1$. If H is dense in itself, then every non-dense set is separable if and only if each non-denumerable set is separable or contains a dense in itself kernel which is dense in a non-empty open set. If H is dense in itself and every non-dense set is separable, then every decreasing family of open sets is of type less than Ω^2. In H_1 a necessary and sufficient condition that $2^{\aleph_0}=\aleph_1$ is that, if the power of $M \geqq \aleph_1$ implies $MM(\aleph_0) \neq 0$, then the power of $M \geqq \aleph_2$ implies $MM(\aleph_2) \neq 0$. It is shown that the following properties are independent: (1) H_1 is condensed in itself; (2) H_1 is locally hypercondensed in itself; (3) every isolated subset of H_1 is at most denumerable. In H_1 property (2) is equivalent to the property that, for each M, $M(\aleph_1) \neq 0$ implies $MM(\aleph_1) \neq 0$.

L. W. Cohen (Lexington, Ky.).

Maximoff, Isaiah. On density points and approximately continuous functions. Tôhoku Math. J. **47**, 237–250 (1940). [MF 4024]

On a closed interval $[a, b]$ let a function f be defined, let (y) $y_1, y_2, y_3, \cdots$ be an ordering of the rational numbers between the greatest lower and least upper bounds of f on $[a, b]$ and let $E_{y_n}\{E^{y_n}\}$ be the set of all points x of $[a, b]$ such that $f(x)<y_n\{f(x)>y_n\}$. Let H^{r+s}, H_{r+s} $(s=0, 1, 2, \cdots; r=1, 2, 3, \cdots)$ be perfect sets and say that the system (H) of these pairs of sets possesses the properties: (a) $(E=lH)$ if $E_{y_r}=\lim_{s\to\infty} H^{r+s}$, $E^{y_r}=\lim_{s\to\infty} H_{r+s}$; (b) $(H \subset E)$ if $H^{r+s} \subset E_{y_r}$, $H_{r+s} \subset E^{y_r}$; (c) $(H \subset sH)_k$ if $H^{r+s} \subset H^{r+s+1}$, $H_{r+s} \subset H_{r+s+1}$ for $r+s+1 \leqq k$ (k finite or infinite); (d) $(H \subset dH)_k$ if for $y_r<y_t$ each point of the set $H^{r+n}\{H_{r+s_1}\}$ is a point of density of the set $H^{t+n}\{H_{t+s_2}\}$, $r+s_1=t+s_2 \leqq k$; (e) $(H \subset yH)_k$ if $H^{r+n} \subset H^{t+n}$, $H_{r+s_1} \subset H_{t+s_2}$ for $y_r<y_t$ and $r+s_1=t+s_2 \leqq k$. The main theorem of the paper is the following: A necessary and sufficient condition that a finite function f of class 1 be approximately continuous is that there exist a system of perfect sets (P) P^{r+s}, P_{r+s} $(s=0, 1, 2, \cdots; r=1, 2, 3, \cdots)$ which enjoy the properties $(E=lP)$, $(P \subset E)$, $(P \subset sP)_\infty$, $(P \subset dP)_\infty$. A very involved but straightforward proof of the necessity of the conditions is given. The author cites an earlier paper of his [Bull. Amer. Math. Soc. **45**, 264–265 (1939)] for a proof of the sufficiency of the conditions.

J. F. Randolph (Ithaca, N. Y.).

Tolstoff, G. Sur la différentielle totale. Rec. Math. [Mat. Sbornik] N.S. **9** (**51**), 461–468 (1941). (French. Russian summary) [MF 4569]

The author constructs two functions $P(x, y)$ and $Q(x, y)$ for which $\partial P/\partial y=\partial Q/\partial x$ at all points of a domain D, but such that there is no function $z(x, y)$ with $\partial z/\partial x=P$, $\partial z/\partial y=Q$. An auxiliary function $f(x, y)$ is introduced which is continuous in $0 \leqq x, y \leqq 1$, with respect to each variable separately, for which the integrals $\int_0^x f(x, t)dt$ and $\int_0^x f(t, y)dt$ are continuous in (x, y) and for which

$$\int_0^1 dx \int_0^1 f(x, y)dy=0, \quad \int_0^1 dy \int_0^1 f(x, y)dx=1.$$

W. Feller (Providence, R. I.).

Macphail, M. S. Functions of bounded variation in two variables. Duke Math. J. **8**, 215–222 (1941). [MF 4586]

R. L. Jeffery has proved [Trans. Amer. Math. Soc. **40**, 171–192 (1937)] that a necessary and sufficient condition for a function $F(x)$ to be of bounded variation in (a, b) is that there exist a sequence of summable functions $s_n(x)$ tending to $F'(x)$ wherever the latter exists, $\int_a^x s_n(x)dx$ tending to $F(x)-F(a)$, and such that $\int_a s_n(x)dx$ is bounded with respect to n and e. In a later paper [Duke Math. J. **5**, 753–774 (1939); cf. these Rev. **1**, 208] the situation is generalized by the same author to two variables: a function $F(x, y)$ is said to be of class V_1 on a rectangle $R(a, b; c, d)$ if there exist a sequence of summable functions $s_n(x, y)$ tending to a limit, with (1) $\int_e s_n(x, y)dxdy$ bounded in n and e;

$$(2) \qquad F(x, y)=\lim_{n\to\infty} \int_{(a, b; x, y)} s_n(x, y)dxdy.$$

Jeffery proved that functions in V_1 are in class H (Hardy

bounded variation). The present author makes the relationship of the two notions clear by showing that, if the left member of (2) is replaced by $F(x, y)-F(x, b)-F(a, y)+F(a, b)$, the class V_1' so defined is the same as class V (Vitali bounded variation), and that class V_1 consists of those functions in H which vanish on the left and lower boundaries of R. The class V_2 was defined by Jeffery as the extension of V_1 obtained by omitting condition (1); he showed that if $F(I)$ is a continuous function of intervals with a finite strong derivative $F_s'(x, y)$, then $\phi(x, y) \equiv F(a, b; x, y)$ is in V_2 with $s_n(x, y)$ tending to $F_s'(x, y)$. Here the unexpected result is obtained that in fact such a sequence of functions $s_n(x, y)$ will exist tending to any arbitrary measurable function, no assumptions being made about differentiability of $F(I)$. The class V_2', formed from V_2 by again modifying condition (2) as above, is shown to coincide with the first two Baire classes. *J. A. Clarkson.*

Laurent Schwartz. Sur les fonctions à variation bornée et les courbes rectifiables. C. R. Acad. Sci. Paris **212**, 331–333 (1941). [MF 4907]

The author gives a proof of the following theorem: if $\varphi(x)$ is continuous and of bounded variation, and $\{P_n(x)\}$ is a sequence of inscribed polygonal functions constructed on a sequence of subdivisions whose norms tend to zero, then in order that the total variation of $\varphi(x)-P_n(x)$ should tend to zero it is necessary and sufficient that $\varphi(x)$ should be absolutely continuous. This theorem was proved in a paper by C. R. Adams and the reviewer [Trans. Amer. Math. Soc. **45**, 322–334 (1939)]. The author asserts, however, that the situation is different if instead of considering the difference $\varphi(x)-P_n(x)$ we consider the "smallest distance" between the two curves. He considers the function $D_n(s)$, the length of the perpendicular dropped from the point M, at a distance s along the curve from one end, to that side of the polygon P_n which subtends the arc containing M. A proof is briefly sketched that the total variation of $D_n(s)$ will tend to zero without the additional assumption that φ is absolutely continuous; moreover, the rectifiable curve considered may be non-planar. *J. A. Clarkson.*

Santaló, L. A. A theorem and an inequality referring to rectifiable curves. Amer. J. Math. **63**, 635–644 (1941). [MF 4689]

Let $V_k(R)$ denote the volume of the k-dimensional sphere of radius R in Euclidean m-space E_m. Let $P=(x_1, \cdots, x_m)$ be a variable point in E_m; $dP=dx_1 \cdots dx_m$. Let $n=n(P)$ denote the number of common points of a Jordan curve C with the surface of the m-dimensional sphere of center P and radius R. Then C is rectifiable if and only if the integral $J=\int n\, dP$ is finite. The length of C then is equal to $J/(2V_{m-1}(R))$. On this theorem is based a definition of the length of a curve. Another application is the following inequality

$$V(R) \leqq L V_{m-1}(R)+V_m(R).$$

Here, L and $V(R)$ denote, respectively, the length of a rectifiable curve in E_m and the volume of the set of points in E_m that have a distance not greater than R from the curve. *P. Scherk* (Bloomington, Ind.).

Best, E. A theorem on Hausdorff measure. Quart. J. Math., Oxford Ser. **11**, 243–248 (1940). [MF 3786]

Let E be a linear set, and $h(t)$ a concave, continuous and strictly increasing function of the real variable t $(0 \leqq t \leqq t_0)$; $h(t)$ is called a dimension function of the set E if the Hausdorff measure of E with respect to the function $h(t)$ is a positive finite number. The author studies sets of a certain type, which he calls sets of type P (they are defined by a partitioning process which is a generalization of the well-known tryadic Cantor-construction), and establishes a sufficient condition that a function $h(t)$ be a dimension function of a given set of type P. Applied to the Cantor set this condition yields the result: the function $h(t)=t^{\log 2/\log 3}$ is a dimension function of the Cantor set. *W. Hurewicz.*

Younovitch, B. Sur la dérivation des fonctions absolument additives d'ensemble. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 112–114 (1941). [MF 4263]

For a space R, a Borel family F of sets in R, a measure μ for elements of F (and extended to an outer measure for sets of R) and a system of neighborhoods $\{A_n(P)\}$, a point P is said to be a point of density of a set E (with respect to the measure and the system of neighborhoods) if

$$\lim_{n\to\infty} \frac{\bar{\mu}[A_n(P)E]}{\bar{\mu}[A_n(P)]}=1.$$

The system of neighborhoods is said to satisfy the law of density if for each measurable set E the measure of E and the measure of its points of density are the same; and is said to be normal if for each set E and $\epsilon>0$ there exists a finite subsystem whose union σ is such that

$$\bar{\mu}(E-E\sigma)=\bar{\mu}(E)-\bar{\mu}(E\sigma)<\epsilon, \quad \Big|\sum_{i=1}^{k} \bar{\mu}[A_{n_i}(P_i)]-\bar{\mu}(E)\Big|<\epsilon.$$

If $\varphi(E)$ is an absolutely additive set function defined on F then the upper derivative

$$\bar{D}_\mu\varphi(P)=\limsup_{n\to\infty} \frac{\varphi[A_n(P)]}{\mu[A_n(P)]},$$

the lower derivative $\underline{D}_\mu\varphi(P)$ and the derivative $D_\mu\varphi(P)$ are defined. The following theorems are stated and then lemmas given on which their proofs are based. (i) In order that a system of neighborhoods $\{A_n(P)\}$ satisfy the law of density (with respect to a function μ), it is necessary and sufficient that the system be normal (with respect to μ). (ii) In order that an absolutely additive set function φ satisfying the condition $|\varphi(E)| \leqq M|\mu(E)|$ should be differentiable with respect to μ, and that the derivative $D_\mu\varphi(P)$ satisfy the condition

$$\varphi(E)=\int_E D_\mu\varphi(P)d\mu(E),$$

it is necessary and sufficient that the system of neighborhoods $\{A_n(P)\}$ satisfy the law of density or, what is the same, be normal with respect to μ. *J. F. Randolph.*

Kershner, Richard. On non-equidistributed averages. Amer. J. Math. **63**, 611–614 (1941). [MF 4685]

This paper deals with representations of Riemann integrals as limits of certain averages. Let there be given a double sequence x_j^r $(j=1, \cdots, n_r;\ r=1, 2, \cdots)$ such that $0 \leqq x_1^r \leqq x_2^r \leqq \cdots \leqq x_{n_r}^r \leqq 1$, and let

$$\phi(x)=\lim_{r\to\infty} (\text{number of values of } j \text{ with } x_j^r \leqq x)/n_r;$$

let $\delta(x)$ be a density function of $\phi(x)$. Then for any R-integrable $g(y)$

$$\lim_{r\to\infty} \frac{1}{n_r} \sum_{j=1}^{n_r} g(x_j^r)/\delta(x_j^r)=\int_0^1 g(y)dy.$$

This formula contains a formula of the reviewer expressing integrals as infinite series [Math. Ann. 110, 718–721 (1935)], and also an identity given by Pólya [Math. Ann. 88, 177 (1923)]. *F. John* (Lexington, Ky.).

Sólyi, A. Über Funktionen, die ein endliches Dirichletsches Integral haben. Acta Univ. Szeged. Sect. Sci. Math. 10, 48–54 (1941). [MF 4431]

Let $g(x_1, x_2, \cdots, x_n)$ be a function of integrable square on the n-dimensional region T, and let the integral

$$\int_T \left\{ \frac{g(x_1, \cdots, x_a+h, \cdots, x_n)-g(x_1, \cdots, x_n)}{h} \right\}^2 dV$$

be bounded uniformly in h. The author states that on account of applications in the calculus of variations it is of interest to know what can be said about the partial derivatives of g, and for what values of ν $\int_T |g|^\nu dV$ is finite. It is shown that the first partial derivatives exist almost everywhere and are of integrable square. The difference quotient converges in mean square to the partial derivative, and g is the Lebesgue integral of its partial derivative. The exponent ν can be arbitrarily great for $n=2$. If $n>2$ the integral converges for values satisfying $\nu<2n/(n-2)$, and there are functions g for which the integral diverges when $\nu>2n/(n-2)$. The case for $\nu=2n(n-2)$ is undecided.

R. L. Jeffery (Wolfville, N. S.).

Romanovski, Paul. Intégrale de Denjoy dans les espaces abstraits. Rec. Math. [Mat. Sbornik] N.S. 9 (51), 67–120 (1941). (French. Russian summary) [MF 4491]

Let R be a regular topological space with the second axiom of denumerability holding. Let e be an element of the Borel sets on R with compact closure, and $\mu(e)$ a non-negative completely additive function of these sets; $\mu(e)$ then serves as a measure function in connection with the notion of summability and Lebesgue integrals $\int_e f\mu(de)$. There is then introduced a class of sets called fundamental sets, denoted by the letters $\mathfrak{A}$, A, a, and satisfying the following axioms: (1) they form a complete system of neighborhoods; (2) their closure is compact; (3) if $A \subset \mathfrak{A}$ there is a decomposition of $\mathfrak{A}$ containing A; (4) the product of two sets possesses a decomposition; (5) all sets admit a decomposition of arbitrarily small measure; (6) if $A=a_1+\cdots+a_n$ and $a_i \subset \mathfrak{A}$ then $A \subset \mathfrak{A}$; (7) all sets $\mathfrak{A}$ are such that there exists A with its frontier in $\mathfrak{A}$ and measure arbitrarily near to that of $\mathfrak{A}$. Let $F(A)$ be additive, $S=a_1+\cdots+a_n$ a decomposition of $\mathfrak{A}$. If, as $\mu(a_i)\to 0$, $\sum F(a_i)\to$ a limit, this limit is the Burkill integral $\int_{\mathfrak{A}} F$ of F over $\mathfrak{A}$. Also $F(A)$ is interior continuous if, for each $A \subset \mathfrak{A}$, $|F(A)-F(a)|<\epsilon$ when $a \subset A$ and $\mu(A-a)<\eta$. A point x of a set of E is isolated in the special sense if there is a neighborhood V containing x with EV the frontier of some set A.

The function f is D-integrable on $\mathfrak{A}$ if there exists $F(A)$ additive and interior continuous such that for any closed set E without special isolated points there exists A containing a point of E with $\int_A F_{\bar{B}} = \int_{\bar{A}\bar{B}} f\mu(de)$ for all $\bar{A} \subset A$, where $F_{\bar{B}}(A)=F(A)$ if $\bar{A}\bar{E}$ is not empty, $F_{\bar{B}}(A)=0$ otherwise. Derivatives of $F(A)$ are defined by means of the ratio $F(A)/\mu(A)$, $\bar{A}$ containing x, and the usual definitions of bounded variation, continuity, absolute continuity and generalized absolute continuity (ACG) are set up for $F(A)$. In order that $F(A)$ be an indefinite D-integral it is necessary and sufficient that $F(A)$ be (ACG), and $D_x F=f$ almost everywhere. Fundamental sets satisfy the condition of mobility if for $\bar{a} \subset \bar{A} \subset \mathfrak{A}$, x on the frontier of A, there exists A' such that $\bar{a} \subset A'$, $\bar{A}' \subset \mathfrak{A}$, $x \in A'$, $\mu(A')=\mu(A)$. If the sets A satisfy this condition, if $F(A)$ is additive and continuous, with upper and lower derivatives finite except on a denumerable set of frontiers of sets A, and if $D_x F$ exists almost everywhere, then $D_x F$ is D-integrable to $F(A)$.

Integrals of the Perron and Ridder types are defined, also major and minor functions based on integrals; for example, $\psi(A)$ is a major to f if for E closed there exists A containing a point of E with $\int_{\bar{A}} \psi_E \geqq \int_{\bar{A}E} f\mu(de)$ for all $\bar{A} \subset A$. This leads to $\bar{D}$-integrals. $F(A)$ has property B if for all closed sets $e \subset \bar{A}$ and on the frontier of A there exists $\bar{A} \subset A$ containing a part of e such that $\bar{A}A$ admits a decomposition S with $\int_a F_e=0$ for all a contained in one of the sets of S. Such functions lead to D'-integrals. Let $\mathfrak{N}$ be an ordered set of elements ν, σ, σ_ν systems of fundamental sets, $\sigma \supset \sigma_\nu$, $\sigma_{\nu_1} \supset \sigma_{\nu_2}$ if $\nu_1 \supset \nu_2$ for each $\nu \in \mathfrak{N}$, then each set σ admits a decomposition in σ_ν. By using decompositions of $\mathfrak{A}$ into σ_ν integrals of variable index ν are obtained. *R. L. Jeffery.*

Romanovski, Paul. Intégrale de Denjoy dans l'espace à n dimensions. Rec. Math. [Mat. Sbornik] N.S. 9 (51), 281–307 (1941). (French. Russian summary) [MF 4554]

This is an application of the idea of integrals of variable index developed in the paper reviewed above, where R is the Euclidean n-space, the set $\mathfrak{N}$ the numbers q on $0<q<1$, σ the set of intervals, σ_q the set of intervals with parameter of regularity not less than q. Examples are given of functions which are D-integrable but not D'-integrable.

R. L. Jeffery (Wolfville, N. S.).

Romanovski, Paul. Intégrale relative à un réseau. Rec. Math. [Mat. Sbornik] N.S. 9 (51), 309–316 (1941). (French. Russian summary) [MF 4555]

In this paper the integrals developed in the papers reviewed above are studied, where the fundamental sets of these papers satisfy the following additional conditions: (1) all fundamental sets are compact; (2) any pair of fundamental sets are disjoint, or one contains the other. The space $\mathfrak{A}$ is covered with sets which would correspond to a system of nets in Euclidean n-space, where each mesh of any net was contained in a mesh of the preceding net. The sets obtained in this way are used as fundamental set.

R. L. Jeffery (Wolfville, N. S.).

Romanovski, Paul. Sur l'existence de l'intégrale de Burkill. Rec. Math. [Mat. Sbornik] N.S. 9 (51), 317–320 (1941). (French. Russian summary) [MF 4556]

In this paper there is a further application of the ideas in the papers reviewed above. Let $S=a_1+\cdots+a_n$ be a decomposition of the n-dimensional interval A into intervals σ_q of parameter of regularity not less than q, $0<q<1$. Moreover, $F(S)=\sum F(a_i)$, and the upper and lower bounds of $F(S)$ as $\mu(a_i)\to 0$ give the upper and lower Burkill integrals of variable index q over A. Let s be a decomposition of the intervals of S into intervals σ_q. Let $I(S)$ be the lower bound of $F(s)$ for fixed S. The upper bound of $I(S)$ is then an upper integral, and the corresponding lower integral is obtained by interchanging the role of upper and lower bounds. Then, obviously

$$\underline{\int}_A^q F \leqq \underline{\int}_A^{*q} F \leqq \bar{\int}_A^{*q} F \leqq \bar{\int}_A^q F,$$

where the first and last are the Burkill integrals. Necessary and sufficient conditions are stated (without proof) for the existence, and for the equality of these integrals.

R. L. Jeffery (Wolfville, N. S.).

Izumi, Shin-ichi. An abstract integral. IV. Proc. Imp. Acad. Tokyo **17**, 1–4 (1941). [MF 4330]

[For the previous notes, cf. these Rev. **1**, 305 and **2**, 258.] Essentially this note gives an abstract setting for the general integral of Daniell [Ann. Math. (2) **19**, 279 (1918)]. **L** is assumed to be a vector lattice, **T** a conditionally σ-complete vector lattice and the index α forms a Moore-Smith set, that is, is subject to a relation ($\geqq$) which is transitive and compositive. Convergence in α is assumed to be either of (1) relative uniform type, (2) order (lattice) type, or (3), if **T** is a Banach space, norm type. A set of transformations T^α on **L** to **T** gives rise to an "integral" via the definition: f of **L** is integrable if $\lim_\alpha T^\alpha f$ exists, where $\lim_\alpha$ is one of the three types mentioned. By assuming that T^α are linear and positive, and possess a certain sequential closure property, while **T** satisfies the condition that for any sequence u_n of **T** there exists a sequence of real numbers λ_n so that $\sum\lambda_n u_n$ converges in the order sense, then the class of integrable functions L has properties comparable to that of Lebesgue integrable functions. A connection with ergodic theory is indicated by assuming $T^n=(1/n)[\tau f+\tau^2 f+\cdots+\tau^n f]$, where τ is on **L** to **L** and τ^n is the nth iterate of τ.

T. H. Hildebrandt (Ann Arbor, Mich.).

Theory of Functions of Complex Variables

Belinfante, M. J. Elemente der intuitionistischen Funktionentheorie. I. Die Cauchyschen Integralsätze und die Taylorsche Reihe. Nederl. Akad. Wetensch., Proc. **44**, 173–185 (1941). [MF 4393]

The author adopts the following definition for regularity of a function $F(z)$ in a region G of the z-plane, which includes uniformity: For every $\epsilon>0$ there is a d such that for all z_1, z_2, z_3 in G, $z_2\neq z_1$, $z_3\neq z_1$ and $|z_2-z_1|<d$, $|z_3-z_1|<d$ imply

$$\left|\frac{F(z_3)-F(z_1)}{z_3-z_1}-\frac{F(z_2)-F(z_1)}{z_2-z_1}\right|<\epsilon.$$

On this basis he derives "intuitionistically" some of the fundamental theorems of complex function theory: in particular, Cauchy's theorem and integral formula, Weierstrass's theorem on double series, and Laurent's expansion, but not, as one would expect, the general residue theorem. With a little caution the classical proofs carry over. No new ideas are required.

H. Weyl (Princeton, N. J.).

Ríos, Sixto. On the analytic continuation of Dirichlet series with infinite maximum density. Revista Union Mat. Argentina **7**, 38–40 (1941). (Spanish) [MF 4456]

The author studies a special kind of overconvergence of one particular series.

S. Mandelbrojt (Houston, Texas).

Curtiss, J. H. On the Jacobi series. Trans. Amer. Math. Soc. **49**, 467–501 (1941). [MF 4341]

The author undertakes a systematic study of the Jacobi series in the following form:

$$(*)\quad a_0+a_1(z-\alpha_1)+\cdots+a_{\lambda-1}(z-\alpha_1)\cdots(z-\alpha_{\lambda-1})+a_\lambda\omega(z)+a_{\lambda+1}\omega(z)(z-\alpha_1)+\cdots.$$

Here $\omega(z)=(z-\alpha_1)\cdots(z-\alpha_\lambda)$ with $\alpha_1,\cdots,\alpha_\lambda$ arbitrary complex. If $F(z)$ is regular in the lemniscate $|\omega(z)|<\mu$ it can be expanded into a uniquely determined series of this type. This classical theorem in connection with the integral representation of the coefficients a_ν is the starting point of the present investigation. First, according to a remark of Walsh, an integral representation is derived for the series (components of $F(z)$) whose terms are the columns of (*). After the substitution $z=z_\rho(w)$, where $z_\rho(w)$ is a proper branch of $\omega(z)=\mu w$, these components become regular functions in $|w|<1$. Further the analogue of the classes H^α of F. Riesz is introduced defined by the condition

$$\int_{|\omega(z)|<\rho\mu}|F(z)|^\alpha|dz|=O(1),\qquad \rho\uparrow 1.$$

It is shown that, if $F(z)$ belongs to H^α on $|\omega(z)|<\mu$, the components defined before belong to certain $H^{\alpha'}$ on the unit circle. An inverse of this theorem is also proved. The rest of the paper is devoted to the analogues of the theorems of Fatou, Parseval, F. and M. Riesz and Young-Hausdorff, and to the discussion of the Fourier expansions of the components on $|w|=1$.

G. Szegö.

Gontcharoff, W. L. et Gontcharoff, M. K. Sur la représentation des fonctions analytiques par des séries des fonctions rationnelles d'un type spécial. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 298–300 (1941). [MF 4384]

The authors announce without proof a number of interesting results regarding expansions of the type

$$(*)\qquad c_0+\sum_{n=1}^{\infty}c_n\left\{\prod_{m=1}^{n}\left(\frac{z-a_m}{1-a_m z}\right)\right\},$$

where $0<a_n\uparrow 1$, $\sum(1-a_n)$ divergent. There exist two circles $u(z)=\Re((z+1)/(z-1))<u_c$ and u_a in which (*) is convergent and absolutely convergent, respectively. Then

$$u_c-u_a\leqq\limsup((\log n)/\textstyle\sum(1-a_m)),\quad 0\leqq m\leqq n.$$

Furthermore, information is obtained about the functions represented by (*). For instance, if $\mu_a<0$ this function is regular for $\mu(z)<\mu_a$. Under the further restriction $n^\sigma(1-a_n)\to\omega$, $0<\sigma\leqq 1$, we have

$$\limsup_{\rho\to 0}\rho^{(1-\sigma)/2\sigma}\log|f(1-\rho e^{-i\psi})|\leqq(2\sigma/(1-\sigma^2))\omega^{1/\sigma}|u_a|^{(1+\sigma)/2\sigma}(2\cos\psi)^{(\sigma-1)/2\sigma},$$

$$\sigma<1;\ 0\leqq\psi<\pi/2.$$

This result needs a modification for $\sigma=1$. Finally it is shown that if $f(z)$ satisfies the increasing condition mentioned, it can be expanded into a series of the type (*).

G. Szegö (Stanford University, Calif.).

Kametani, Shunji. Boundary values of analytic functions. Proc. Imp. Acad. Tokyo **17**, 60–64 (1941). [MF 4453]

Using a method similar to that of S. Kakeya [Proc. Imp. Acad. Tokyo **13**, 292–295 (1937)], the author proves a theorem from which he deduces as corollaries: (i) the sufficiency of the condition of Privaloff [J. École Polytech. **24**, 77–112 (1924); the author's reference to this paper is incorrect] that a Cauchy integral over a certain rectifiable Jordan contour C should tend (from the inside) non-tangentially to a limit almost everywhere on C; (ii) the well-known theorem of Fatou on the limit values of a function bounded

in $|z|<1$. The author's method constitutes a simplification of proof of the essential part of Privaloff's result.

D. C. Spencer (Cambridge, Mass.).

Verjbinsky, M. An extension of Laguerre's theorem on the roots of a transcendent function. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 781–784 (1941). [MF 4476]

The zeros of the three entire functions $\sum a_n z^n$, $P(z)$ and $\sum a_n P(n) z^n$ are connected by certain relations governing their location, the first of which were discovered by Laguerre. The relations discussed by the author are not very different from those already considered in the literature. [Compare, for example, his theorem 3 with Jensen, Acta Math. **36**, 190 (1913).] *G. Pólya* (Providence, R. I.).

Obrechkoff, Nikola. Sur les fonctions entières limites de polynomes dont les zéros sont réels et entrelacés. Rec. Math. [Mat. Sbornik] N.S. **9 (51)**, 421–428 (1941). (French. Russian summary) [MF 4564]

The limit of a sequence of polynomials whose roots are all real is necessarily an entire function, of genus not exceeding two, of a well characterized class. The author completes this well-known result by answering the following question: What are the conditions under which two functions of this class, without common zeros, are the limits of two polynomials each of which has real zeros separating the zeros of the other? The analogous question for polynomials with positive real roots is also treated.

G. Pólya (Providence, R. I.).

Pólya, G. Sur l'existence de fonctions entières satisfaisant à certaines conditions linéaires. Trans. Amer. Math. Soc. **50**, 129–139 (1941). [MF 4871]

The author proves an existence theorem for entire functions, which is a generalization of results due to Whittaker, Poritsky and Gontcharoff. He considers a periodic sequence $a_0, a_1, a_2, \cdots$ of points in the complex plane with period p. That is, $a_{n+p}=a_n$, $n=0, 1, 2, \cdots$. A sequence of integers $\alpha_0, \alpha_1, \alpha_2, \cdots$ is also considered. Here $\alpha_{n+p}=\alpha_n+p$, $n=0, 1, 2, \cdots$; $0\leqq\alpha_0\leqq\alpha_1\leqq\alpha_2\leqq\cdots$. If $a_k=a_l$ and $k<l$ then $\alpha_k<\alpha_l$. Further there is given a sequence $A_0, A_1, A_2, \cdots$ such that

$$\lim_{n\to\infty} |A_n/\alpha_n!|^{1/\alpha_n}=0.$$

The author then asks if there exists an entire function $F(z)$ such that $F^{(\alpha_k)}(a_k)=A_k$, $k=0, 1, 2, \cdots$. He shows that there is such a function if there exists no polynomial $P(z)$ of degree less than p (and not identically zero), which satisfies the p equations $P^{(\alpha_k)}(a_k)=0$, $k=0, 1, 2, \cdots, p-1$.

N. Levinson (Cambridge, Mass.).

Gelfond, A. On the coefficients of periodic functions. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 95–98 (1941). (Russian. English summary) [MF 4508]

The author proves the following theorem: Let $f(z)$ be an entire periodic function of finite order ρ. Let $f(z)=\sum_0^\infty a_n z^n$. Let $N(n)$ be the number of coefficients a_k, $0\leqq k\leqq n$, which are different from zero, and let $d=\overline{\lim}_{n\to\infty} N(n)/n$. Then $d\geqq 1/2\rho$. The proof makes use of the auxiliary function $\phi(z)$. Since by a change of scale the period of $f(z)$ can be made 1, $f(z)$ may be represented by a Fourier series $f(z)=\sum_{-\infty}^{\infty} A_k e^{2\pi i k z}$. The auxiliary function is defined by

$$\phi(z)=A_0+e^{(\pi i/2)z}\sum_1^\infty A_k k^z+e^{-(\pi i/2)z}\sum_1^\infty A_{-k}k^z.$$

Since $a_n=(2\pi)^n\phi(n)/n!$, it follows that if $a_n=0$ then $\phi(n)=0$. Thus the more a_n that vanish the more zeros $\phi(n)$ must have on the positive real axis. Thus making use of the relationship between the growth of an analytic function in a sector and its number of zeros, the author proves his theorem by showing that, if $d<1/2\rho$, $\phi(z)\equiv 0$.

N. Levinson (Cambridge, Mass.).

Selberg, Atle. Über ganzwertige ganze transzendente Funktionen. Arch. Math. Naturvid. **44**, 45–52 (1941). [MF 4424]

Let $f(z)$ be an entire function and call $M(r)$ the maximum of $f(z)$ in the circle $|z|\leqq r$. If $f(z)$ takes rational integral values for $z=0, 1, 2, \cdots$ and $\limsup_{r\to\infty} r^{-1}\log M(r)\leqq a$, where $a=\log 2+(1/1500)$, then $f(z)$ is necessarily of the form $P(z)2^z+Q(z)$, where $P(z)$ and $Q(z)$ are polynomials. This remarkable result of the author cannot be found in the foregoing papers on the subject written by the reviewer, G. H. Hardy, E. Landau and F. Carlson. The author's proof follows an essentially new line; it is based on the consideration of the successive differences of the series of successive differences $f(0), \Delta f(0), \Delta^2 f(0), \cdots$.

It may be observed that the result, except for the numerical value of a, can also be obtained by a different proof. It follows from the hypothesis concerning $M(r)$ that the power series $\sum f(n)z^{-n}$ is regular outside the domain $(\log r)^2+\phi^2\leqq a^2$ (we put $z=re^{i\phi}$). But, if $a>\log 2$ and a is sufficiently near to $\log 2$, the transfinite diameter of this domain is less than 1, and therefore the power series, having integral coefficients, represents a rational function whose poles are at $z=1$ and $z=2$; hence the result. Even the best value of a could be discussed. *G. Pólya* (Providence, R. I.).

Milloux, H. Sur une nouvelle extension d'une inégalité de M. R. Nevanlinna. J. Math. Pures Appl. (9) **19**, 197–210 (1940). [MF 4620]

The second fundamental inequality of R. Nevanlinna for meromorphic functions majorizes the characteristic index $T(r, f)$ of a meromorphic function $f(z)$ by means of the sum of three indices of density relative to the zeros of f and of $f-1$ and to the poles of f, and by a complementary term in general negligible in comparison with $T(r, f)$. The author in a forthcoming book extends the inequality by means of the indices of density relative to the zeros and poles of f, and to the zeros of $\psi-1$, in place of the zeros of $f-1$, where

$$\psi(z)\equiv\sum_0^l \alpha_i(z)f^{(i)}(z),$$

the coefficients $\alpha_i(z)$ being holomorphic in the domain of holomorphism of $f(z)$.

In this paper the author has a new method by which he gets rid of the poles of f and compares $T(r, f)$ with the indices of density of $f-a$, $f-b$ and $\psi-c$, where a, b, c are three finite constants. The author solves this problem principally in the case where $a=0$, $b=c=1$, and in certain cases for general finite constants a, b, c. When $\psi(z)\equiv f'(z)$ the author obtains the theorem: Let $f(z)$ be meromorphic in a domain containing $|z|=\rho>r$. Then

$$T(r, f)<3N\left(r, \frac{1}{f}\right)+2N\left(r, \frac{1}{f-1}\right)+N\left(r, \frac{1}{f'-1}\right)+S(r)$$

with

$$S(r)=44\overset{+}{\log}\,T(\rho,f)+33\log\frac{\rho}{\rho-r}+22\log\frac{\rho}{r}+11\overset{+}{\log}\frac{1}{\rho}$$
$$+3\overset{+}{\log}\,|f(0)|-\log|f'(0)|+330$$
$$+2\overset{+}{\log}\,|\phi(0)|-\log|\phi(0)-1|-\log|\phi'(0)|,$$

where $\phi(z)\equiv(1-f')/f^2$. Here it is supposed that the origin is not a zero of f, $1/f$, $f-1$, f', ϕ, $\phi-1$ and ϕ'.

If ψ replaces f' the author obtains the more general theorem: Let $f(z)$ be meromorphic in a domain D containing $|z|=\rho$. Let $\psi\equiv\sum_0^l\alpha_i\cdot f^{(i)}$ ($l\geqq 1$). The functions α_i are holomorphic in D; α_0-1 and α_l never vanishing. Then

$$T(r,f)<(2l+1)N\left(r,\frac{1}{f}\right)$$
$$+(l+1)N\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{\psi-1}\right)+S(r)$$

with

$$S(r)=a(l)\overset{+}{\log}\,T(\rho,f)+(l+2)\sum_0^l m(\rho,\alpha_i)$$
$$+m\left(\rho,\frac{1}{\alpha_0-1}\right)+a(l)\log\frac{\rho}{\rho-r}+a(l)\log\frac{\rho}{r}$$
$$+a(l)\overset{+}{\log}\frac{1}{\rho}+a(l)+(2l+1)\overset{+}{\log}\,|f(0)|-l\log|f'(0)|$$
$$+2\overset{+}{\log}\,|\phi(0)|-\log|\phi'(0)|-\log|\phi(0)-1|,$$

where

$$\phi\equiv\frac{1-\psi}{f^{l+1}}\frac{1}{1-\alpha_0},\qquad \rho>r.$$

$a(l)$ is a positive constant depending only upon l. The origin is not a zero of f, $1/f$, $1-f$, f', $\psi-1$, $\phi-1$ and ϕ' but these restrictions are not fundamental to the methods used. The author shows that the restriction $\alpha_0\not\equiv 1$ is, however, fundamental. *M. S. Robertson* (New Brunswick, N. J.).

Laurent Schwartz, Mme. Exemple d'une fonction méromorphe ayant des valeurs déficientes non asymptotiques. C. R. Acad. Sci. Paris **212**, 382–384 (1941). [MF 4933]

The author exhibits a meromorphic function for which zero and infinity are deficient values but which are not asymptotic values for the function. *M. S. Robertson.*

Ahlfors, Lars V. The theory of meromorphic curves. Acta Soc. Sci. Fennicae. Nova Ser. A. **3**, no. 4, 31 pp. (1941). [MF 4484]

A system of $n+1$ entire functions $x_i=x_i(t)$ ($i=0, 1, \cdots, n$) is said to define a "meromorphic curve" in a complex $(n+1)$-dimensional vector space E^{n+1}. The object of this paper is to develop for such systems of functions an analogue of R. Nevanlinna's theory of meromorphic functions. A program of this kind was undertaken by H. and J. Weyl [Ann. of Math. (2) **39**, 516–538 (1938)]. The author obtains a series of relations which he terms the first and second laws of equidistribution. These relations include as special cases the results of H. and J. Weyl and Borel's theorem on systems of n meromorphic functions.

The curve with components $x_i(t)$ ($i=0, 1, \cdots, n$) is denoted by $x=x(t)$; the kth derivative of $x(t)$ is denoted by $x^{(k)}(t)$. The multivectors $[xx'\cdots x^{(k-1)}]$ are denoted by X^k ($k=1, \cdots, n+1$). It is assumed that $x=x(t)$ lies in no lower linear dimensional subspace of E^{n+1}; E^k is used to denote an k-dimensional linear subspace and an k-vector of absolute value one lying in this subspace. The contraction of X^k and E^k is denoted by (X^kE^k), the absolute value of (X^kE^k) by $|X^kE^k|$. Further, $N_k(r, E^k)=\int_{r_0}^r n_k(r, E^k)dr/r$, where $n_k(r, E^k)$ is the number of zeros of $(X^kE^k):|X^k|$ in $|t|\leqq r$; and $N_k(r)=\int_{r_0}^r n_k(r)dr/r$, where $n_k(r)$ is the number of zeros of X^k in $|t|\leqq r$. With these conventions the first law of equidistribution may be stated in the form of the following system of identities:

$$(\mathrm{I}_k^h)\qquad m_k(r, E^h)-m_k(r_0, E^h)+T_k(r, E^h)+N_k(r, E^h)=T_k(r),$$

which prevail for every linear subspace E^h of E^{n+1} and $k=1, \cdots, n$. Here

$$m_k(r, E^h)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{|X^k|}{|X^kE^h|}d\varphi,$$

$$T_k(r)=\left|\int_{r_0}^r\frac{1}{r}\frac{1}{2\pi}\int_0^{2\pi}\log|X^k|d\varphi-N_k(r),\right.$$

$$T_k(r, E^h)=\begin{cases}\left|\int_{r_0}^r\frac{1}{r}\frac{1}{2\pi}\int_0^{2\pi}\log|X^kE^h|d\varphi-N_k(r, E^h)-N_k(r),\right. & k\neq h,\\ 0, & k=h.\end{cases}$$

These relations are obtained for $k=h$ by the argument principle in the same manner as Ahlfors obtained the first fundamental relation of Nevanlinna [Soc. Sci. Fenn. Comment. Phys.-Math. **8**, no. 10 (1935)]. For $k\neq h$ these relations are obtained from the case $k=h$ by a set of averaging processes. Introducing the ω notation of H. and J. Weyl [loc. cit.], the author shows that, if one of the $T_k(r)$ is of finite order, then all the $T_k(r)$ are of the same order. By the use of weighted averages, where the weight functions have appropriate singularities, the second law of equidistribution is obtained. This states that for a finite system of E^h in general position (that is, no h-dimensional linear complex C^h contains more than $\binom{n+1}{h}-1$ of the E^h from the system)

$$(\mathrm{II}_k^h)\qquad \sum m_k(r, E^h)=\binom{n+1}{k}\bar{T}_k(r)$$
$$-\sum_{i=k-h}^{k}\binom{n-i}{k-i}\binom{i-1}{h-k+i}\bar{T}_i(r)-\binom{k}{h-1}N_{n+1}(r)+\omega(T^2)$$

for $k\geqq h$; and

$$(\mathrm{II}_k^h)\qquad \sum m_k(r, E^h)$$
$$=\binom{n+1}{k}\bar{T}_k(r)-\sum_{i=k}^{n+1-h+k}\binom{i-1}{k-1}\binom{n-i}{h-k-1}\bar{T}_i(r)+\omega(T^2)$$

for $k\leqq h$. Here $\bar{T}_k(r)=T_k(r)+N_k(r)$ and

$$T(r)=\max_{(k=1,\cdots,n)}T_k(r).$$

The second law of equidistribution is applied to the class of exponential curves $x_i=e^{\lambda_i a}$, where the λ_i are $n+1$ constants distinct from one another. *M. H. Heins.*

***van Kuik, Jan. Continuous Iteration.** Thesis, University of Utrecht, 1940. viii+73 pp. (Dutch)

Let $w(z)$ be analytic, with positive real part, in the half plane $\Re(z)>0$. A continuous analogue of the sequence of successive iterates of $w(z)$ is the function $z=z_t(z_0)$ satisfying the differential equation $dz/dt=w(z)$ (t is a real variable) and reducing to z_0 when $t=0$. The function z_t has been studied by Wolff [Compositio Math. **6**, 296–304 (1938)]; this thesis is devoted to a more detailed study. The behavior of z_t differs according to whether $\lambda=\lim_{z\to\infty}w(z)/z$ is posi-

tive or zero. In case $\lambda>0$, corresponding to each value α with positive real part there is a function $k(\alpha)=\lim_{t\to\infty} x_t/|\alpha_t|$, which is studied in detail. Let $T(z_0)$ denote the locus of points z_t for fixed z_0 and variable t. If $\lambda>0$, Wolff showed that $x_t=\Re(z_t)\to\infty$ as $t\to+\infty$ on each $T(z_0)$; and $\arg z_t\to\varphi(z_0)$, where $\varphi(z_0)$ is harmonic and takes all values between $-\pi/2$ and $\pi/2$. On the other hand, if $\lambda=0$, x_t may approach a finite limit ($T(z_0)$ then has a vertical asymptote); $\arg z_t$ may fail to approach any limit, but, if it approaches a limit on one curve $T(z)$, it approaches the same limit on all. The author discusses various possible behaviors of $\arg z_t$, and illustrates them by examples. He gives new necessary and sufficient conditions for $T(z_0)$ to have a vertical asymptote. Special behavior of z_t for special classes of functions $w(z)$ is discussed. The behavior of the curves $T(z_0)$ as $t\to-\infty$ is also discussed and illustrated by examples. *R. P. Boas, Jr.*

Wolff, J. Théorème sur l'itération d'une représentation conforme. II. Nederl. Akad. Wetensch., Proc. **44**, 195–197 (1941). [MF 4395]

This note is a continuation of two earlier notes of the author [C. R. Acad. Sci. Paris **210**, 658–659 (1940); Nederl. Akad. Wetensch., Proc. **43**, 1016–1017 (1940); see these Rev. **2**, 83 and 186] which we denote (in order of appearance) by (i) and (ii). Here the author shows (by similar methods) that the result of (ii) remains true assuming only that the frontier of D is a Jordan curve extending to infinity in the two directions $\arg z=\pm\pi/2$; therefore that the additional restriction on the frontier used in (ii) is unnecessary. Combining this result with the theorem proved in (i), the author points out that $\lambda>0$ is a necessary and sufficient condition that N should contain a domain of angular width π at infinity. *D. C. Spencer* (Cambridge, Mass.).

Tsuji, Masatsugu. A theorem on conformal representation. Jap. J. Math. **17**, 97–108 (1940). [MF 4606]

The author demonstrates the following theorem: Let $C_n (n=1, 2, \cdots)$ and C be rectifiable Jordan curves in the w-plane, $w=0$ being in the interior of C_n $(n=1, 2, \cdots)$ and C. Further, let $C_n\to C$ in the sense of Fréchet, and $L_n\to L$ as $n\to\infty$, where L_n is the length of C_n $(n=1, 2, \cdots)$ and L is the length of C. If the interior of the unit circle in the z-plane is mapped one-to-one and conformally onto the interior of C_n by $w=f_n(z)$ $(f_n(0)=0; f_n'(0)>0)$ $(n=1, 2, \cdots)$ and onto the interior of C by $w=f(z)$ $(f(0)=0; f'(0)>0)$, then

$$\lim_{n\to\infty}\int_0^{2\pi}|f'(e^{i\theta})-f_n'(e^{i\theta})|\,d\theta=0.$$

The proof is based upon a theorem of F. and M. Riesz [Math. Z. **18**, 87–95 (1923)], which the author demonstrates in detail. The fact that $f_n'(z)$ $(n=1, 2, \cdots)$ and $f'(z)$ do not vanish for $|z|<1$ is exploited systematically. This permits one to express $f_n'(z)$ $(n=1, 2, \cdots)$ and $f'(z)$ as the squares of functions which are analytic and single-valued for $|z|<1$. The author gives an application of his theorem to the theory of approximation of analytic functions by polynomials. *M. H. Heins* (Princeton, N. J.).

Tumura, Yosiro. Sur les théorèmes de M. Valiron et les singularités transcendantes indirectement critiques. Proc. Imp. Acad. Tokyo **17**, 65–69 (1941). [MF 4454]

Generalizing some theorems of Valiron [Rend. Circ. Mat. Palermo **46** (1925); C. R. Acad. Sci. Paris **200**, 713–715 (1935)] the author proves that, if a meromorphic algebroid f has k branches, and if $\underline{\lim}\,[T(r, f)/(\log r)^2]<\infty$, then this function has not more than k distinct asymptotic values. Some applications. *S. Mandelbrojt* (Houston, Tex.).

Valiron, Georges. Division en feuillets de la surface de Riemann définie par $w=(e^z-1)/z+h$. J. Math. Pures Appl. (9) **19**, 339–358 (1940). [MF 4628]

For a given entire function $w=f(z)$ of finite order the subdivision of the z-plane into domains of univalence for $f(z)$, bounded by curves $\arg w=$ const. of constant argument, is obtained by a process due to Iversen. Radial continuation of those branches of the inverse function $z=\phi(w)$ which are holomorphic at the origin yields stars of holomorphism and these constitute a first system of sheets. Similarly, the branches which admit $w=0$ as algebraic critical point can be decomposed into branches which are uniform in the neighborhood of the origin, and radial prolongation yields a second system of sheets. A third system comes from the branches which have transcendental singularities at $w=0$. Consider the function $w=(e^z-1)/z+h$. Anomalous situations can arise as follows. Iversen has shown that for $h=0$ there is an incomplete domain of univalence, corresponding to a sheet of opening less than 2π. For $h=-1$, the author has shown that there is a singular domain: the values $w=f(z)$ corresponding to the domain and its frontier do not cover a line l. The author now shows that for $h=+1$ there is the further anomaly of an improper subdivision of the z-plane: let l be a line in the z-plane whose points are limit points of the frontiers of an infinite set of domains of univalence; for this subdivision, it is impossible to join a point of one domain which is on one side of l to a point of another domain which lies on the other side of l by means of a continuous curve which does not cut an infinitude of the domains of univalence.

E. F. Beckenbach (Ann Arbor, Mich.).

Basoco, M. A. On the inflexional elastica. Amer. Math. Monthly **48**, 303–309 (1941). [MF 4514]

After a brief historical introduction, the author gives a solution of the famous problem of the elastic rod with hinged ends, somewhat different from the classical treatment, as given, for example, in Appell et Lacour, Fonctions Elliptiques, in terms of maximum deflection and initial angle [the transformation (5) looks mysterious without reference to the general theory!]. The value of this problem as an exercise in the use of elliptic functions has long been recognized. The details on page 309 look unnecessarily complicated on account of the introduction of new notations, p and $K(p)$, for the magnitudes denoted by k' and K' on page 308. The possible shapes are not discussed and no indication is given of the conditions under which points of inflection occur. *A. Dresden* (Swarthmore, Pa.).

Wintner, Aurel. On Riemann's fragment concerning elliptic modular functions. Amer. J. Math. **63**, 628–634 (1941). [MF 4688]

In the first part of his posthumous fragment on the limiting values of the elliptic modular functions [B. Riemann, Mathematische Werke, 2nd ed., pp. 455–461], Riemann gives some examples of functions which are represented by trigonometric series but are discontinuous everywhere dense. It seems that Riemann did not prove the convergence of his series, and until now no proof of the validity of his

formulae had been published. The author shows that Riemann's statements are simple consequences from known results in the modern theory of Fourier series.

C. L. Siegel (Princeton, N. J.).

***Bergmann, Stefan. Sur les fonctions orthogonales de plusieurs variables complexes avec les applications à la théorie des fonctions analytiques.** Interscience Publishers, Inc., New York, 1941. 62 pp. $1.50.

Das vorliegende Heft gibt eine Uebersicht über die wichtigsten Eigenschaften und Anwendungen der Systeme von Orthogonalfunktionen zweier komplexen Veränderlichen z_1, z_2 ($z_k = x_k + iy_k$), und der mit ihrer Hilfe definierten Kernfunktion. In einem einleitenden Kapitel werden die Begriffe der analytischen Funktion, der analytischen Fläche, der Hyperfläche und die einiger spezieller Bereiche des R_4 definiert und beschrieben. Kapitel II bringt dann die Hauptsätze der Theorie der Orthogonalfunktionen: Eine Folge $\{\varphi^{(\nu)}\}$ von in einem Bereiche $\mathfrak{B}$ analytischen Funktionen heisst ein normiertes Orthogonalsystem, falls das Volumenintegral $\int_{\mathfrak{B}} \varphi^{(\nu)}\overline{\varphi^{(\mu)}}d\omega = 0$ für $\nu \neq \mu$ ($d\omega = dx_1 dy_1 dx_2 dy_2$) und $\int_{\mathfrak{B}} \varphi^{(\nu)}\overline{\varphi^{(\nu)}}d\omega = 1$. Der Ausdruck

$$\sum_{\nu=1}^{\infty} \varphi^{(\nu)}(z_1, z_2) \cdot \overline{\varphi^{(\nu)}(t_1, t_2)}$$

ist der Kern des Systems; er ist analytisch in z_1, z_2, t_1, t_2 für (z_1, z_2) und (t_1, t_2) aus $\mathfrak{B}$. Für eine gegebene in $\mathfrak{B}$ quadratintegrierbare Funktion f (das heisst, f ist in $\mathfrak{B}$ analytisch und $\int_{\mathfrak{B}} f\bar{f}d\omega < \infty$) werden die Grössen $a_\nu = \int_{\mathfrak{B}} f\overline{\varphi^{(\nu)}}d\omega$ die Fourierkoeffizienten von f in bezug auf $\mathfrak{B}$ und das System $\{\varphi^{(\nu)}\}$ genannt. Falls für jede Funktion f einer Klasse F der Ausdruck

$$\int_{\mathfrak{B}} |f - \sum_{\nu=1}^{n} a_\nu \varphi^{(\nu)}|^2 d\omega$$

beliebig klein gemacht werden kann (n genügend gross), heisst $\{\varphi^{(\nu)}\}$ ein in bezug auf $\mathfrak{B}$ und die Klasse F geschlossenes System; in diesem Falle konvergiert für jedes f aus F die Reihe $\sum a_\nu \varphi^{(\nu)}$ in jedem ganz in $\mathfrak{B}$ gelegenen Teilbereich absolut und gleichmässig gegen f. Es gilt nun der fundamentale Satz, dass es zu jedem Bereich $\mathfrak{B}$ und zu der Klasse $E(\mathfrak{B})$ aller in $\mathfrak{B}$ quadratintegrierbaren Funktionen ein geschlossenes Orthogonalsystem gibt, das bei Vorgabe eines Bezugspunktes (t_1, t_2) eindeutig bestimmt ist. Der Kern eines in bezug auf $\mathfrak{B}$ und die Klasse $E(\mathfrak{B})$ geschlossenen Systems ist nur von $\mathfrak{B}$ abhängig; er wird die Kernfunktion $K_{\mathfrak{B}}(z_1, z_2; \bar{t}_1, \bar{t}_2)$ von $\mathfrak{B}$ genannt. Für ein gegebenes (t_1', t_2') ist $[K_{\mathfrak{B}}(t_1', t_2'; \bar{t}_1', \bar{t}_2')]^{-1}$ das Minimum aller $\int_{\mathfrak{B}} f\bar{f}d\omega$ mit f aus $E(\mathfrak{B})$ und $f(t_1', t_2') = 1$; das Minimum wird durch die ebenfalls eindeutig $\mathfrak{B}$ zugeordnete Minimalfunktion

$$M_{\mathfrak{B}}^{1}(z_1, z_2; t_1, t_2) = K_{\mathfrak{B}}(z_1, z_2; \bar{t}_1, \bar{t}_2)[K\ (t_1, t_2; \bar{t}_1, \bar{t}_2)]^{-1},$$
$$(t_1, t_2) = (t_1', t_2'),$$

erreicht. Für einige spezielle Bereiche (Kreisbereiche und gewisse verallgemeinerte Sternbereiche) kann ein geschlossenes Orthogonalsystem und damit auch die Kernfunktion explizit angegeben werden; ein allgemeines Verfahren gibt es bisher nicht.

Im Kapitel III werden Minimalprobleme der folgenden Art behandelt: Sei $\alpha_{q\nu}$, $\nu = 1, 2, \cdots$; $q = 1, \cdots, n$, ein System komplexer Zahlen mit $\sum |\alpha_{q\nu}|^2 < \infty$, $q = 1, \cdots, n$, und $X_1, X_2, \cdots, X_n$ gegebene komplexe Zahlen. Dann wird diejenige (eindeutig bestimmbare) Funktion $f^{(n)}(z_1, z_2)$ konstruiert, die den Bedingungen

$$\sum_{\nu=1}^{\infty} \alpha_{q\nu} \int_{\mathfrak{B}} f^{(n)}\overline{\varphi^{(\nu)}}d\omega = X_q, \qquad q = 1, \cdots, n,$$

genügt und $\int_{\mathfrak{B}} f\bar{f}d\omega$ zu einem Minimum macht. Bei besonderen Voraussetzungen über die $\alpha_{q\nu}$ kann man beweisen, dass für zwei Bereiche $\mathfrak{B}_1$ und $\mathfrak{B}_2$ mit $\mathfrak{B}_1 \subset \mathfrak{B}_2$ auch $\min \int_{\mathfrak{B}_1} f\bar{f}d\omega < \min \int_{\mathfrak{B}_2} f\bar{f}d\omega$ (es ist dies eine gewisse Verallgemeinerung des Schwarzschen Lemmas). Mit Hilfe der bei der Lösung von Minimalproblemen dieser Art benutzten Methoden wird eine notwendige und eine hinreichende Bedingung aufgestellt, damit eine in $\mathfrak{B}$ analytische Funktion f mit $\int_{\mathfrak{B}} (\log |f|)^2 d\omega < \infty$ eine gewisse vorgegebene Reihe von Funktionen $f^{(\nu)}$ als Nullstellen besitzt. Ebenso wird eine obere und untere Schranke für eine in $\mathfrak{B}$ meromorphe Funktion mit einem in $\mathfrak{B}$ "vollständigen" System von Null- und Polstellenfunktionen gegeben.

Im letzten Kapitel werden kurz die Grundzüge einer gegenüber analytischen Abbildungen invarianten Metrik beschrieben, die mit Hilfe der Kernfunktion eingeführt wird und deren Bogenelement durch die Hermitesche Form: $ds^2 = J \sum T_{m\bar{n}}^{(\mathfrak{B})}(z_1, z_2; \bar{z}_1, \bar{z}_2) dz_m d\bar{z}_n$ gegeben ist, wobei

$$T_{m\bar{n}}^{(\mathfrak{B})} = \frac{\partial^2 \log K_{\mathfrak{B}}(z_1, z_2, \bar{z}_1, \bar{z}_2)}{\partial z_m \partial \bar{z}_n}$$

und J eine willkürliche, gegenüber analytischen Abbildungen invariante positive Funktion ist. Für $J \equiv 1$ ist die absolute Invariante dieser Metrik $I = K_{\mathfrak{B}} \cdot (T_{1\bar{1}}^{(\mathfrak{B})} T_{2\bar{2}}^{(\mathfrak{B})} - |T_{1\bar{2}}^{(\mathfrak{B})}|^2)^{-1}$ stets positiv und die Krümmung in jeder analytischen Richtung kleiner 2. Das genauere Studium der Metrik behält sich der Verfasser für ein zweites Heft vor. [Zu einem Teil der behandelten Probleme vgl. auch Behnke-Thullen: Theorie der Funktionen mehrerer komplexen Veränderlichen, Ergebnisse der Mathematik und ihrer Grenzgebiete, v. III, 3, Springer, Berlin, 1934; vor allem die Seiten 105–109.]

P. Thullen (Quito).

Oka, Kiyosi. Sur les domaines pseudoconvexes. Proc. Imp. Acad. Tokyo **17**, 7–10 (1941). [MF 4332]

Die Pseudokonvexität ist seit langem als eine der fundamentalen (notwendigen) Eigenschaften der Regularitätsbereiche (genauen Existenzbereiche) analytischer Funktionen von n komplexen Veränderlichen bekannt. Ob auch umgekehrt jeder pseudokonvexe Bereich notwendig ein Regularitätsbereich ist, war eines der wichtigsten offenen Probleme, das bisher nun für wenige spezielle Bereiche und meist unter sehr einschränkenden Bedingungen gelöst war. Die vorliegende Arbeit kündet nun die Lösung des Problems in einer sehr allgemeinen Form an: Jeder endliche, schlichte pseudokonvexe Bereich im Raume von zwei komplexen Veränderlichen ist ein Regularitätsbereich. Verfasser behält sich für eine nächste Arbeit den vollständigen Beweis vor und gibt hier nur eine kurze Darstellung eines grundlegenden Hilfssatzes.

P. Thullen (Quito).

Schapiro, Z. Sur l'existence des représentations quasi-conformes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 690–692 (1941). [MF 4472]

The author investigates the question of the existence of quasi-conformal representations for plane regions in the case in which the characteristics are not necessarily continuous.

W. T. Martin (Cambridge, Mass.).

Reade, Maxwell and Beckenbach, E. F. Generalizations to space of the Cauchy and Morera theorems. Trans. Amer. Math. Soc. **49**, 354–377 (1941). [MF 4337]

Consider a surface $S: x_j=x_j(u, v)$, $j=1, 2, 3$, where the functions x_j are continuous in a simply connected domain D. If the functions x_j possess continuous first partial derivatives which satisfy the relations (1) $E-G=F=0$, where $E(u, v)$, $F(u, v)$, $G(u, v)$ are the coefficients of the first fundamental form, then S is said to be given in isothermic representation. Using an order evaluation for the expression

$$\sum_{j=1}^{3}\left[\int_{C_\rho} x_j(u, v)dz\right]^2, \qquad z=u+iv,$$

where C_ρ is the circle $|z-z_0|=\rho$, and using an alternative form $\sum_1^3(\lambda x_j)^2=0$ $(\lambda=\partial/\partial u+i\partial/\partial v)$ for (1), the authors derive several sets of necessary and sufficient conditions for the isothermal mapping of D onto a surface S, where in some cases the surface S is required to possess certain properties. The first theorem states: If the functions $x_j(u, v)$ have continuous first partial derivatives in a simply connected domain D, then a necessary and sufficient condition that they map D isothermally on a surface S is that, for each point $z_0=u_0+iv_0$ of D,

$$\sum_{j=1}^{3}\left[\int_{C_\rho} x_j(u, v)dz\right]^2=o(\rho^4).$$

The second theorem gives a characterization of isothermic spherical maps that do not map circles on circles. The third gives a characterization of minimal surfaces in isothermal representation and of those isothermic spherical maps that map circles on circles. The fourth (and final) theorem gives a characterization of isothermic plane maps. Various corollaries are obtained. *W. T. Martin* (Princeton, N. J.).

Isaacs, Rufus P. The finite differences of polygenic functions. Bull. Amer. Math. Soc. **47**, 444–448 (1941). [MF 4535]

The author shows how the values of the difference quotients of a polygenic function, formed at a given point, may be represented by the points of a surface in three space. He studies the geometry of this surface, one of whose plane sections is the Kasner derivative circle. *P. Franklin.*

Fourier Series and Integrals, Theory of Approximation

Erdös, P. On a conjecture of Steinhaus. Univ. Nac. Tucumán. Revista A. **1**, 217–220 (1940). [MF 4064]

If all partial sums $\sum_{-N}^{N}a_ne^{inx}$, $N=0, 1, 2, \cdots$, of a Fourier series are non-negative, then there exists a quantity $\varphi(c)$ for $c>0$, such that the number of $|a_n|\geqq c$ for $|n|\leqq N$ is $O((\log N)^{\varphi(c)})$. This includes Sidon's result that $\sum_{-N}^{N}|a_n|=o(N)$, but does not yet confirm Steinhaus' conjecture that $a_n=o(1)$. *S. Bochner* (Princeton, N. J.).

Nikolsky, S. M. Estimations of the remainder of Féjèr's sum for periodical functions possessing a bounded derivative. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 210–214 (1941). [MF 4831]

Consider the class $\mathfrak{M}$ of functions of period 2π, having $r-1$ continuous derivatives, the last of them satisfying the Lipschitz condition

$$|f^{(r-1)}(x)-f^{(r-1)}(y)|\leqq K|x-y|.$$

Here K denotes a constant, the same for the whole class. It is shown that, if $\sigma_n(x; f)$ denote the Fejér means of the Fourier series of f, then $\max|f(x)-\sigma_n(x; f)|$, for $0\leqq x\leqq 2\pi$ and $f\in\mathfrak{M}$, is equal to

$$\frac{2K}{\pi}\frac{\log n}{n}+O(n^{-1}) \qquad \text{for } r=1,$$

and to

$$\frac{4K}{\pi(n+1)}c_r+O\left(\frac{\log n}{n^r}\right) \qquad \text{for } r>1.$$

Here c_r denotes the number $\sum_{\nu=0}^{\infty}(-1)^{\nu(r-1)}/(2\nu+1)^r$. The result holds for certain means analogous to Fejér means and connected with the theory of trigonometric interpolation. *A. Zygmund* (South Hadley, Mass.).

Nikolsky, S. M. An asymptotic estimation of the remainder under approximation by interpolating trigonometric polynomials. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 215–218 (1941). [MF 4832]

Let $U_n(x; f)$ denote the (Lagrange) trigonometric polynomial coinciding with the function f at the $2n+1$ equidistant points $2\pi i/(2n+1)$, $i=0, 1, \cdots, 2n$. Let $E_n(x; \mathfrak{M})$ denote the upper bound of $|f(x)-U_n(x; f)|$ for f belonging to a class $\mathfrak{M}$ of functions f of period 2π. It is shown that, if $\mathfrak{M}$ is the class of functions satisfying the Lipschitz condition

$$|f(x')-f(x')|\leqq K|x'-x''|^\alpha, \quad 0<\alpha\leqq 1; \ K \text{ fixed},$$

then $E_n(x; \mathfrak{M})$ is equal to

$$\frac{K}{\pi^{1-\alpha}}\frac{\log n}{n^\alpha}|\sin(n+\tfrac{1}{2})x|+O(n^{-\alpha}).$$

If $\mathfrak{M}$ is the class of functions having $r-1$ derivatives, and such that $|f^{(r-1)}(x'')-f^{(r-1)}(x')|\leqq K|x''-x'|$, then $E_n(x; \mathfrak{M})$ is equal to

$$\frac{8}{\pi^2}A_r\frac{K}{n^r}\log n|\sin(n+\tfrac{1}{2})x|+O(n^{-r}),$$

uniformly with respect to x. Here

$$A_r=\sum_{\nu=0}^{\infty}(-1)^{\nu(r-1)}/(2\nu+1)^{r+1}, \quad r=1, 2, \cdots.$$

A. Zygmund (South Hadley, Mass.).

Salem, Raphaël. The absolute convergence of trigonometrical series. Duke Math. J. **8**, 317–334 (1941). [MF 4596]

The paper gives a number of results concerning the absolute convergence of trigonometric series. In particular, the following theorems are established. (i) Suppose that the trigonometric series

$$(*) \qquad \sum_{n=1}^{\infty}\rho_n\cos(nx-\alpha_n), \qquad \rho_n\geqq 0,$$

has coefficients ρ_n non-increasing. Then, if the series (*) converges in more than one point of the interval $0\leqq x<\pi$, the series $\rho_1+\rho_2+\cdots$ converges. (ii) A point set E is said to be of type N if there is a trigonometric series (*) converging absolutely in E but not everywhere. If $F(x)$ is any non-decreasing function constant in the intervals contiguous to a perfect set E of type N, then

$$\overline{\lim_{n\to\infty}}\int_0^{2\pi}\cos 2nx\,dF(x)=F(2\pi)-F(0).$$

(iii) A point set is said to be of type N' if there is a trigono-

metric series (*) such that

$$\rho_1+\rho_2+\cdots=\infty, \quad \sum_{n=1}^{\infty}\rho_n\cos^2(nx-\alpha_n)<\infty$$

in E. The class of sets N' coincides with the class of sets N. (iv) Suppose that $\rho_1+\rho_2+\cdots=\infty$. If $\alpha<\frac{1}{2}\pi$, the set of points x where

$$(**) \qquad \overline{\lim_{n\to\infty}}\,\Big|\sum_{\nu=1}^{n}\rho_\nu\cos(\nu x-\alpha_\nu)\Big|\Big/\sum_{\nu=1}^{n}\rho_\nu\leqq\alpha$$

is of measure 0. If $\rho_n=O(1)$, the ratio on the left of (**) tends to $\frac{1}{2}\pi$ for almost every x. *A. Zygmund.*

Bosanquet, L. S. The absolute Cesàro-summability problem for differentiated Fourier series. Quart. J. Math., Oxford Ser. **12**, 15–25 (1941). [MF 4667]

In a previous paper [Quart. J. Math., Oxford Ser. **10**, 67–74 (1939)] the author showed that a necessary and sufficient condition for a differentiated Fourier series of a function $f(t)$ to be summable $(C, \alpha+1)$ $(\alpha\geqq 0)$ is that (i) the function $\{f(x+t)-f(x-t)\}/(4\sin\frac{1}{2}t)$ should be integrable in the Cesàro-Lebesgue sense over the interval $0\leqq t\leqq\pi$ and (ii) its cosine Fourier series should be summable (C, α) at the point $t=0$. The author now shows that the theorem holds if we replace in it Cesàro summability by absolute Cesàro summability of the same order and replace Cesàro-Lebesgue integrability by absolute Cesàro-Lebesgue integrability (a new notion which the author introduces in this paper). *A. Zygmund* (South Hadley, Mass.).

Wang, Fu-Traing. A note on the summability of lacunary partial sums of Fourier series. Quart. J. Math., Oxford Ser. **12**, 57–60 (1941). [MF 4672]

It is shown that, if $s_n(x)$ is the nth partial sum of the Fourier series of a function $f(x)\varepsilon L$, then the sequence $\{s_{n^2}(x)\}$ is summable (C, α), $\alpha>0$, to $f(x)$ at almost every point x. For $\alpha=1$ this result was obtained by Zalcwasser [Studia Math. **6**, 82–88 (1936)]. *A. Zygmund.*

Plancherel, M. Quelques remarques sur les transformations de Fourier des fonctions de plusieurs variables. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 20–26 (1940). [MF 4399]

In section 1 the author extends the theorem of M. Riesz on conjugate functions to functions of several variables and uses this to extend results of Hille, Offord and Tamarkin [Bull. Amer. Math. Soc. **39**, 768–774 (1933) and **41**, 427–436 (1935)] concerning Fourier transforms of functions of one variable to functions of several variables. In section 2 he extends results of Kolmogoroff and Marcinkiewicz concerning convergence of partial sums of Fourier series of functions in L_2 to Fourier transforms. For the case of two variables the result reads: If $f(x, y)\varepsilon L_2^2$ and if the positive sequences $\{a_n\}$, $\{b_n\}$ are such that $a_{n+1}/a_n>\lambda>1$, $b_{n+1}/b_n>\lambda>1$, $a_n/b_n<\lambda'$, $b_n/a_n<\lambda'$, then

$$F(x, y; a_n, b_n)=\frac{1}{2\pi}\int_{-a_n}^{a_n}\int_{-b_n}^{b_n}f(\xi, \eta)e^{-i(x\xi+y\eta)}d\xi d\eta$$

tends to the Fourier transform of $f(x, y)$ for almost all (x, y). *E. Hille* (Stanford University, Calif.).

Laurent Schwartz. Sur le module de la fonction caractéristique du calcul des probabilités. C. R. Acad. Sci. Paris **212**, 418–421 (1941). [MF 4912]

The main result of the note is that the characteristic function $\Pi_{n=1}^{\infty}\cos(\pi(p^n/q^n)t)$ tends to 0 with $1/t$, if $p=2, 3, \cdots, q-1$, and the fraction p/q is irreducible. This is known [see Nina Bary, Rec. Math. [Mat. Sbornik] (N.S.) **2** (**44**), 699–722 (1937) and J. Kershner, Amer. J. Math. **58**, 450–452 (1936)]. *A. Zygmund* (South Hadley, Mass.).

Guinand, A. P. General transformations and the Parseval theorem. Quart. J. Math., Oxford Ser. **12**, 51–56 (1941). [MF 4671]

A function $f(x)$ in $0<x<\infty$ is said to belong to the class S_k^2, $k\geqq 0$, if there exists a function $f^{(k)}(x)$ such that

$$f(x)=(1/\Gamma(k))\int_x^{\infty}(t-x)^{k-1}f^{(k)}(t)dt$$

almost everywhere and $x^k f^{(k)}(x)$ belong to the Lebesgue class $L^2(0, \infty)$. The author observes that the cosine transform and, more generally, any Watson transform of a function belonging to S_k^2 belongs to the same class, and this is an extension of the theorem of Plancherel-Watson which refers to S_0^2. Also S_k^2 includes S_r^2 for $0\leqq r\leqq k$.

S. Bochner (Princeton, N. J.).

San Juan, Ricardo. Caractérisation de la transformation de Laplace par la loi du produit ou règle de la "Faltung." Portugaliae Math. **2**, 91–92 (1941). [MF 4279]

Enzo Levi [Atti Accad. Naz. Lincei. Rend. **24**, 422–426 (1936)] has characterized the Laplace transformation $L[\phi(t)]=\int_0^{\infty}\phi(t)e^{-zt}dt$ among bounded linear operations on $\phi(t)$ by the relation (1) $L[\phi'(t)]=zL[\phi(t)]-\phi(0)$. This relation is now shown to be implied by the law of convolution. However, in deriving (1) from that law, the author uses the relation (2) $L(1)=1/z$, without showing how the law of convolution implies (2). *I. J. Schoenberg.*

Chlodovsky, I. Certaines propriétés interpolatoires des fonctions absolument monotones de deux variables. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 387–390 (1940). [MF 3605]

Let $F(x, y)$ be absolutely monotone for $x\leqq 0$, $y\leqq 0$. The interpolation problems treated are as follows: (1) Find an absolutely monotone $F(x, y)$ such that $F(-k, y)=f_k(y)$ $(k=0, 1, 2, \cdots; y\leqq 0)$, where the $f_k(y)$ are given. (2) Similarly, if $F(-k, -p)=C_{k,p}$ are given $(k=0, \cdots, n; p=0, \cdots, m$, where n is finite and $m\leqq\infty)$. The existence is settled in terms of certain conditions on the data and the uniqueness cases are completely characterized. Theorem 2 may be found in a paper by the reviewer [Trans. Amer. Math. Soc. **35**, 476 (1933)]. *I. J. Schoenberg* (Philadelphia, Pa.).

Krein, M. Sur le problème du prolongement des fonctions hermitiennes positives et continues. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 17–22 (1940). [MF 3541]

It is here shown for the first time that a function $f(x)$ which is positive definite in the interval $-A<x<A$ may be extended to a function which is positive definite for $-\infty<x<\infty$. A different proof is due to D. Raikov [the same C. R. **26**, 860–865 (1940); cf. these Rev. **2**, 190]. The author's proof is very simple and based on the possibility of extending a linear positive operation from a certain linear space to a more inclusive space. Beyond this result the author states five theorems and several lemmas which seem to exhaust the important question of the unicity of the positive definite extension of $f(x)$ from $(-A, A)$ to $(-\infty, \infty)$. The results are analogues of the work of Hamburger, R. Nevanlinna and M. Riesz for the problem of moments. Since only the main points of the proofs are

sketched, a discussion of this theory should await its detailed publication. *I. J. Schoenberg* (Philadelphia, Pa.).

Krein, M. and Lewitan, B. On some minimum-problems in the class of Stepanoff almost periodic functions. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **17**, 111–124 (1940). (English. Russian summary) [MF 4753]

Let real numbers $(0\leqq)\lambda_1<\lambda_2<\cdots<\lambda_n$ and a_k, b_k $(k=1, 2, \cdots, n)$ be given (b_1 being zero if λ_1 is zero), and let $\mathfrak{M}$ denote the mean of an almost periodic function. The paper deals with the following two problems: (I) To find among all real Stepanoff almost periodic functions $\varphi(t)$ which satisfy

$$\mathfrak{M}\{\varphi(t)\exp(i\lambda_k t)\}=a_k+ib_k, \quad k=1, \cdots, n,$$

the one which makes $\sup|\varphi(t)|$ a minimum, and to find the value γ of this minimum. (II) To find the trigonometric polynomials

$$T(t)\equiv\sum(\xi_k\cos\lambda_k t+\eta_k\sin\lambda_k t)$$

satisfying $\sum(a_k\xi_k+b_k\eta_k)=1$ which make $\mathfrak{M}|T(t)|$ a minimum, and to find the value μ of this minimum. It is shown that these two problems are closely related. In fact $\gamma=\mu^{-1}$, the desired $\varphi(t)$ and $T(t)$ exist, $\varphi(t)$ is essentially unique, and $\varphi(t)=\gamma\operatorname{sgn}T(t)$ almost everywhere. These results are generalized by replacing trigonometric functions by Stepanoff functions, and certain special cases are noted. It appears to the reviewer that in the first displayed line of Theorem 5 the equality sign should be replaced by a minus sign, and in the second displayed line the inequality sign should be reversed. *R. H. Cameron* (Cambridge, Mass.).

Lewitan, B. Correction to the paper: "New generalization of the almost-periodic functions of H. Bohr." Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **17**, 125–126 (1940). (Russian) [MF 4754]

The paper in question appeared in the same journal (4) **15**, no. 2, 3–34. See also Ann. of Math. (2) **40**, 805–815 (1939); these Rev. **1**, 53.

Kovanko, A. S. Sur les systèmes compacts de fonctions presque périodiques généralisées de W. Stepanoff. Rec. Math. [Mat. Sbornik] N.S. **9 (51)**, 389–401 (1941). (French. Russian summary) [MF 4562]

This is a more extended presentation of a paper published by the author in the C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 211–213 (1940) [these Rev. **2**, 190].

R. H. Cameron (Cambridge, Mass.).

Hartman, Philip and Wintner, Aurel. On the convexity of averages of analytic almost periodic functions. Amer. J. Math. **63**, 581–583 (1941). [MF 4681]

The authors prove the following theorem: if a function $g=g(\sigma, t)$ defined on the strip $\alpha\leqq\sigma\leqq\beta$; $-\infty<t<+\infty$ is non-negative, subharmonic and uniformly less than $C_1|t|^{C_2}$ and has for each σ an average $M_t\{g(\sigma, t)\}$ which is approached uniformly with respect to translations t, then it follows that $M_t\{g(\sigma, t)\}$ is a convex function of σ on $\alpha\leqq\sigma\leqq\beta$. *R. H. Cameron* (Cambridge, Mass.).

Bržečka, V. Sur les polynomes orthogonaux dans deux intervalles symetriques. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **17**, 75–97 (1940). (Russian. French summary) [MF 4750]

The author considers polynomials orthogonal on $E=[-1, -\beta]+[\beta, 1]$ $(0\leqq\beta<1)$ with the weight-function $q(x)=|x-\gamma|t(x)$ $(|\gamma|\leqq\beta; t(x)\equiv t(-x))$, in particular,

$$q(x)=\left(\frac{x+\beta}{x-\beta}\right)^r\left(\frac{1-x}{1+x}\right)^s, \quad \left(\frac{x+\beta}{x-\beta}\right)^r(1-x^2)^s; \quad r, s=\pm\tfrac{1}{2}.$$

In the latter case all is readily reduced to trigonometric polynomials, which yields the explicit form of these polynomials, their recurrence relation, differential equation, etc. Finally, making use of known results of Szegö and Bernstein (related to the weight-function $t(x)/(1-x^2)^{\frac{1}{2}}$), the author discusses the expansion of an analytic function in series of orthogonal polynomials corresponding to the weight-function

$$\frac{|x-\gamma|t(x)}{((1-x^2)(x^2-\beta^2))^{\frac{1}{2}}}, \quad |x-\gamma|\left(\frac{1-x^2}{x^2-\beta^2}\right)^{\frac{1}{2}}t(x),$$

$$|x-\gamma|\left(\frac{1-x}{(1+x)(x^2-\beta^2)}\right)^{\frac{1}{2}}t(x),$$

$t(x)$ being properly specified. *J. A. Shohat.*

Geronimus, J. On polynomials orthogonal with regard to a given sequence of numbers. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **17**, 3–18 (1940). (Russian. English summary) [MF 4744]

The author continues the study of the polynomials

$$P_r(z)=\begin{vmatrix} & & & c_n \\ & \Delta_n & & c_{n+1} \\ & & & \vdots \\ & & & c_{2n-1} \\ 1 & z\cdots & z^{n-1} & z^n \end{vmatrix}, \quad \Delta_n=|c_{i+j}|_{i,j=0}^{n-1}\neq 0, \quad n=0, 1, 2, \cdots; \Delta_0=1,$$

orthogonal with respect to a given sequence of numbers $\{c_n\}$ in the sense of $\sigma(P_m(z)P_n(z))=0$, $m\neq n$, where $\sigma(\sum_{i=0}^n\alpha_i z^i)\equiv\sum_{i=0}^n\alpha_i c_i$. In the present paper we have the solution of the following problems. (I) Find the sequence $\{c_n'\}$ with respect to which the polynomials

$$\sigma\left\{\frac{P_n(x)-P_n(z)}{x-z}\right\}\equiv R_{n-1}(x), \quad n=1, 2, \cdots,$$

are orthogonal. (II) Find necessary and sufficient conditions that the sequence $\{P_n(z)+A_nP_{n-1}(z)\}$ $(n=0, 1, 2, \cdots; P_{-1}\equiv 0)$ be orthogonal with respect to a sequence $\{c_n^*\}$, $\Delta_n^*\neq 0$ $(n=1, 2, \cdots)$. The author gives explicit relations between $\{c_n\}$, $\{c_n'\}$, $\{c_n^*\}$. He studies in detail the important case where $c_n=\int_{-\infty}^{\infty}x^n d\psi(x)$ $(n=0, 1, \cdots)$, $\psi(x)\uparrow$ in $(-\infty, \infty)$. Here he discusses $\psi'(x)$, $\psi^*(x)$. *J. A. Shohat.*

Geronimus, J. Sur quelques propriétés des polynomes orthogonaux généralisés. Rec. Math. [Mat. Sbornik] N.S. **9 (51)**, 121–135 (1941). (Russian. French summary) [MF 4492]

The author considers polynomials $\{P_n(z)\}$ orthogonal with respect to a given sequence of complex numbers $\{c_n\}$ [cf. the preceding review] such that

$$\Delta_n=\begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{-1} & c_0 & \cdots & c_{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{-n+1} & c_{-n+2} & \cdots & c_0 \end{vmatrix}\neq 0, \quad n=1, 2, \cdots; c_{-k}=\bar{c}_k; \quad k=0, 1, 2, \cdots,$$

the orthogonality meaning $\sigma\{P_n(z)\bar{P}_m(1/z)\}=0, m\neq n$, where $\sigma(z^k)=c_k$ $(k=0, \pm 1, \pm 2, \cdots)$. In the special case

$$(*) \quad c_n=\frac{1}{2\pi}\int_0^{2\pi}e^{in\theta}d\sigma(\theta), \quad n=0, 1, 2, \cdots; \sigma(\theta)\uparrow \text{ in } [0, 2\pi],$$

$\{P_n(z)\}$ are the ordinary orthogonal polynomials (OP) in

the complex domain. The author shows that $\{P_n(z)\}$ have many properties of the ordinary OP in the real domain (that is, on a segment); namely, analogous recurrence relation and associated continued fraction, its numerators, functions of the second kind [all this is given explicitly]. The case (*) is studied in detail. *J. A. Shohat.*

Geronimus, J. On the orthogonality of a system of polynomials on several contours. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **16**, 12–32 (1940). (Russian. English summary) [MF 4729]

The author considers the problem of polynomials orthogonal on several contours. He makes use of the functions of second kind, also of Faber's polynomials and of some known properties of power series with a finite number of distinct coefficients. In this way he obtains results extending those of Szegö, Walsh and Merriman. *J. A. Shohat.*

Rémès, E. J. Sur les approximations par les moyennes d'ordre $2k$ et celles d'après le principe des moindres carrés. Rec. Math. [Mat. Sbornik] N.S. **9 (51)**, 437–450 (1941). (Russian. French summary) [MF 4566]

The author gives a method for constructing effectively the polynomial of best approximation to a given continuous function, in the sense of the least $2k$th powers (k positive integer). Generalizing the classical setting, we wish to minimize

$$(1)\qquad \int_E \rho(x)[p_n(x)-f(x)]^{2k}d\mu(e) \equiv \int_E \rho(x)[\sum_{i=0}^{n} c_i u_i(x)-f(x)]^{2k}d\mu(e).$$

Here E is a given set of points in a properly characterized abstract space, with measure $\mu(e)$, $\rho(x)$ (positive and measurable), $f(x)$ and $u_i(x)$ are given functions, $p_n(x)=\sum_{i=0}^{n}c_iu_i(x)$ (c_i=const.) is the desired "polynomial." The method consists in minimizing

$$(2)\qquad \int_E \rho(x)[P_n(x)-f(x)]^{2k-2}[Q_n(x)-f(x)]^2d\mu(e),$$

where P_n is any given polynomial and Q_n is the desired polynomial. The latter problem requires the solution of a system of linear algebraic equations. It is shown that the minimum of (1) may be approximated by adding to P_n in (2) a correction of the form $\alpha(Q_n-P_n)$, and that the infinite sequence P_n, $P_n+\alpha(Q_n-P_n)\equiv P_{n1}$, $P_{n1}+\alpha_1(Q_{n1}-P_{n1})\equiv P_{n2}$, $\cdots$ tends to the minimizing polynomial $p_{n0}(x)$ of (1).

J. A. Shohat (Philadelphia, Pa.).

Mirakyan, G. Approximation des fonctions continues au moyen de polynômes de la forme $e^{-nx}\sum_{k=0}^{m}C_{k,n}x^k$. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 201–205 (1941). [MF 4828]

Starting with the readily provable relation

$$\lim_{n\to\infty}\sum_{k=0}^{m}\frac{n^kx^k}{k!}e^{-nx}=1 \text{ uniformly in } [0,r']\text{; } 0<r'<r=\varliminf_{n\to\infty} m/n,$$

and using considerations customarily employed in problems of this kind, the author shows that, for any function $f(x)$ continuous in $[0, r']$,

$$\lim_{n\to\infty}\sum_{k=0}^{m}\frac{n^kx^ke^{-nx}}{k!}f\left(\frac{k}{n}\right)=f(x)$$

uniformly in $[0, r']$. He next shows that if $|f(x)-\Phi(x)|<\epsilon$ in $[a, b]$ ($\epsilon>0$ arbitrary) then, for $a\leqq x\leqq b$,

$$|f(x)-F(x)|\equiv|f(x)-\Phi(x)\sum_{k=1}^{p}\frac{f(x_k)}{\Phi(x_k)}l_k(x)|<\eta(\epsilon),\ \lim_{\epsilon\to 0}\eta(\epsilon)=0.$$

Here p is bounded, $a\leqq x_1<x_2<\cdots<x_p\leqq b$, and $\{l_k(x)\}$ are the fundamental polynomials of the Lagrange interpolation formula. Evidently $F(x_k)=f(x_k)$, $k=1, 2, \cdots, p$, so that we have here a connection between the theory of approximation and that of interpolation for continuous functions.

J. A. Shohat (Philadelphia, Pa.).

Bernstein, S. Sur l'approximation d'une fonction continue par un opérateur linéaire différentiel d'un polynôme. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 15–42 (1941). (Russian. French summary) [MF 4268]

The author considers the differential operator

$$(1)\qquad D_k(\ \ )\equiv\sum_{i=0}^{k}\varphi_i(x)\,Y^{k-i}(x),$$

where $\varphi_i(x)$ are continuous on a given interval (a, b), and the differential equation (2) $D_k(Y)=A(x)$, where $A(x)$ is continuous on (a, b). The operator $D_k(Y)$ is said to belong to the class S ($\bar{S}$) if the sequence of functions $f_n(x)\equiv D_k(x^n)$, $n=0, 1, \cdots$, is complete (incomplete) on (a, b). Thus, $D_k(Y)\varepsilon S$ means that to any $A(x)$ continuous on (a, b) and to any $\epsilon>0$ there corresponds a polynomial $P_\epsilon(x)$ such that

$$|D_k(P_\epsilon(x))-A(x)|<\epsilon, \qquad a\leqq x\leqq b.$$

While $D_k(Y)\varepsilon S$, if $\varphi_0(x)\neq 0$ on (a, b), here the author discusses the case where $\varphi_0(x)$ vanishes on a certain set $E\subset(a, b)$, and gives conditions and criteria under which $D_k(Y)\varepsilon S$. Also, in case $D_k(Y)\varepsilon\bar{S}$, he studies the functions which we must add to $\{f_n(x)\}$ in order to make this sequence complete. The discussion is based on a special definition of "regular" solutions for (2), and also on the behavior of the solutions of (2) near the "singular points" of (1), that is, points belonging to E. *J. A. Shohat* (Philadelphia, Pa.).

Differential Equations

Gleyzal, André. On the equation $dy/dx=f(x, y)$. Bull. Amer. Math. Soc. **47**, 254–256 (1941). [MF 4169]

Sufficiency conditions on $f(x, y)$ have been variously given for the existence of a one-parameter family of solution curves of the differential equation $dy/dx=f(x, y)$ simply covering a given open region R. Pointing out that corresponding necessary conditions are lacking in the literature, the author proves the following theorem: A necessary condition for the existence of a one-parameter family of solution curves $y=\phi(x, a)$, a the parameter, of the differential equation $dy/dx=f(x, y)$ simply covering a given open region R is that f be of Baire's first class. With the aid of Baire's theorem, this conclusion follows from easy consequences of the assumed hypotheses. *H. Blumberg.*

Huruya, Sigeru. Rayleighsches Prinzip und Weinsteins Einschliessungssatz. Mem. Fac. Sci. Kyūsyū Imp. Univ. A. **1**, 209–211 (1941). [MF 3981]

The author considers a differential system consisting of a linear homogeneous self-adjoint differential equation $L(\phi(s))=\lambda p(s)\phi(s)$, and self-adjoint boundary conditions

$\Re(\phi(s))=0$; the function $p(s)$ is assumed to be positive and continuous. He then shows that a theorem of Weinstein on the location of characteristic values for this system is a consequence of Rayleigh's principle for an associated self-adjoint system. *W. T. Reid* (Chicago, Ill.).

Infeld, L. On a new treatment of some eigenvalue problems. Phys. Rev. (2) **59**, 737–747 (1941). [MF 4304]

A technique is developed which yields the solutions of certain boundary value problems of second order with great ease. The method, as far as developed, applies to a class of Sturm-Liouville problems containing two parameters, with density functions which are positive in the fundamental intervals; that is, to eigen-problems of mathematical physics, such as have a line spectrum, with a finite number (or "ladder") of eigen-solutions belonging to each finite order. By copious examples which comprise the oscillating rotator and the generalized Kepler problem treated both by Schrödinger's equation and by Dirac's equations as well as in a spherical space, it is shown that all the important quantum-mechanical eigenvalue problems can be treated by the method in a straight-forward manner. "This seems to suggest that it is something more than merely a mathematical trick. Unfortunately, I was unable to find a deeper reason for this." [Quoted from the end of the paper.]

The method consists in factorization of the second order differential expression containing the "ladder" parameter into two mutually adjoint first order operators. A simple criterion is derived as to when and how an equation can be factorized. The method is applicable if factorization is possible and if the resulting right side which contains the other parameter (taking on the eigenvalues of infinite progression) satisfies a positivity condition. Bessel's equation is an example where factorization is possible, but the second condition is not fulfilled. In cases where the method is applicable, all the eigen-solutions are obtained almost without effort, that is, by solving a first order equation given by the first order operator and then applying a recurrence formula with the adjoint operator, which yields one finite "ladder" of solutions. *H. G. Baerwald* (Cleveland, Ohio).

Sintzov, D. M. Über die Kongruenz der Integralkurven des Normalsystems. II. Typen der singulären Punkte. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **18**, 1–25 (1940). (Russian. German summary) [MF 4758]

The singularities of the integral curves of the system $dx:dy:dz=P:Q:R$ have been investigated by Poincaré. This paper takes up the same question starting with the equation of a "connex" $\sum P_i u_i=0, i=1, 2, 3, 4$, of order m and class 1, and its osculating bilinear connex $\sum(\partial_k P_i)X_k U_i=0$, $i, k=1, 2, 3, 4$. The cases studied are given by the equation $dx_i/dt=\sum a_{ik}x_k$, and the classification of singularities depends on the behavior of the roots of the secular equation (δ_{ik} Kronecker symbol) Det $|a_{ik}-\lambda\delta_{ik}|=0$. All cases are enumerated and illustrated by examples. *D. J. Struik.*

Janet, Maurice. Sur le problème de Pfaff. J. Math. Pures Appl. (9) **19**, 307–318 (1940). [MF 4626]

Given the Pfaffian form $\omega=X_i dx^i$, $i=1, 2, \cdots, n$. The equations

$$(\partial X_i/\partial x_k-\partial X_k/\partial x_i)dx_k=0$$

determine systems of curves called (A), which do not change when a total differential is added to ω. The equations

$$(\partial X_i/\partial x_k-\partial X_k/\partial x_i)dx_k=\lambda X_i, \quad X_i dx^i=0,$$

determine systems of curves called (B), which do not change when ω is multiplied by an arbitrary factor. It is shown that both systems of equations are completely integrable, and a distinction is made between the cases that (A) do and do not satisfy the equation $\omega=0$. The curves (A) are Goursat's "extrémales" [see Bull. Soc. Math. France **44**, 13 (1916)]. *D. J. Struik* (Cambridge, Mass.).

Mitrinovitch, Dragoslav S. Sur l'intégration d'une équation linéaire aux dérivées partielles. C. R. Acad. Sci. Paris **210**, 783–785 (1940). [MF 4891]

The paper gives the general solution of the following partial differential equation $z_n+\alpha_1 z_{n-1}+\cdots+\alpha_n z=0$, where the α_i are constants and

$$z_k=x^k\frac{\partial^k z}{\partial x^k}+\binom{k}{1}x^{k-1}y\frac{\partial^k z}{\partial x^{k-1}\partial y}+\cdots+y^k\frac{\partial^k z}{\partial y^k}.$$

F. G. Dressel (Durham, N. C.).

Haag, Jules. Sur certaines équations aux dérivées partielles. C. R. Acad. Sci. Paris **212**, 259–261 (1941). [MF 4897]

In terms of the notation in the above review, the paper reduces the problem of finding the general solution of any partial differential equation of the type $H(x, y, z, z_1, \cdots, z_n)=0$ to the problem of finding the general solution of an ordinary differential equation. *F. G. Dressel.*

Pfeiffer, G. Sur les équations, les systèmes d'équations aux dérivées partielles du premier ordre à plusieurs fonctions inconnues, qui possèdent une intégrale de S. Lie généralisée. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 195–197 (1941). [MF 4825]

A generalized integral of S. Lie of rank ρ and class $n-1$ is a system

$$\Phi_i(\phi_1, \cdots, \phi_\rho, \phi_{\rho+1}, \cdots, \phi_{\rho+n-1})=0, \quad i=1, 2, \cdots, \rho,$$

where $\phi=\phi(z_1, \cdots, z_k, x_1, \cdots, x_n)$ and $\rho\leqq k$. Among the system of ρ equations

$$M_i+\sum_{\sigma\tau}^{k\,n} M^{\sigma}{}_{i\tau}p_{\sigma\tau}+(H)=0, \quad p_{\sigma\tau}=\partial z_\sigma/\partial x_\tau$$

with general integral Φ_i there are $\sigma=k-\rho+1$ Jacobian systems $\sum a_\tau p_{k\tau}-C_k=0, \tau=1, 2, \cdots, n$. The integration of each of them is equivalent to the integration of a linear homogeneous equation with one unknown. These equations lead to a system of linear homogeneous equations with integrals $\phi_1, \cdots, \phi_\rho, \phi_{\rho+1}, \cdots, \phi_{\rho+n-1}$. A second theorem deals with the case of class less than $n-1$. *D. J. Struik.*

Lusin, N. Un cas du théorème de Janet-Riquier. I. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 5–8 (1941). [MF 4506]

Let S denote a system of differential equations, and A a set of initial conditions for S. The author suggests that one may determine whether or not $S+A$ is compatible, if S can be reduced to the normal form of Janet-Riquier. As an illustration, the following system of differential equations is treated:

$$\sum_{j=1}^{n}(a_{ij}x_j''+b_{ij}x_j'+c_{ij}x_j)=f_i(t), \quad i=1, \cdots, n,$$

where the primes mean differentiation with respect to t and a_{ij}, b_{ij}, c_{ij} are constants. *F. G. Dressel* (Durham, N. C.).

Lusin, N. Un cas du théorème de Janet-Riquier. II, III. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 419–424 (1941). [MF 4836]

[Cf. the preceding review.] The author proves several theorems concerning the following system of linear differential equations by reducing it to the normal form of Janet-Riquier:

$$(1) \qquad \sum_{j=1}^{n} a_{ij}x_j = \delta_i f(t), \quad x_i(0) = x'(0) = 0\,; \; i = 1, \cdots, n,$$

where a_{ij} is a polynomial with constant coefficients and of second degree in D, $D = d/dt$, $\delta_1 = 1$, $\delta_2 = \cdots = \delta_n = 0$. His main result is: if the system (1) has a solution for every analytic function $f(t)$, and the determinant $|a_{ij}|$ is not zero while the minor of a_{11} is zero, then x_1 is zero. For the sufficiency part of the theorem on page 422 to hold it must be added that the ranks of the matrices of the left members of the first and second systems are $m-1$ and $n-1$, respectively. *F. G. Dressel* (Durham, N. C.).

Mendes, Marcel. Sur une équation aux dérivées partielles du second ordre. C. R. Acad. Sci. Paris **212**, 112–114 (1941). [MF 4893]

The author writes down the general integral of

$$(1) \qquad \Delta_2 u = u_{xx} + u_{yy} + u_{zz} - u_{xy} - u_{xz} - u_{yz}.$$

He also associates the distance element

$$(2) \qquad ds^2 = dx^2 + dy^2 + dz^2 - dxdy - dxdz - dydz$$

with (1) and makes some comments on the space defined by (2). The author should have pointed out that the relation of (1) to (2) is strictly formal, since the usual properties of transformations brought about by such an association do not hold here. *F. G. Dressel* (Durham, N. C.).

Lowan, Arnold N. On the problem of wave-motion for the wedge of an angle. Philos. Mag. (7) **31**, 373–381 (1941). [MF 4699]

Given the partial differential equation

$$(*) \qquad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = \Omega(r, \theta, t),$$

where $\Omega(r, \theta, t)$ is a prescribed function of the indicated arguments. The author solves this equation under the following conditions: (i) the limits as $t \to 0$ of $u(r, \theta, t)$ and $\partial u(r, \theta, t)/\partial t$ are assigned functions of r and θ; (ii) $u(r, \theta, t)$ is an assigned function of r and t for $\theta = 0$ and $\theta = \theta_0$ (that is, along the rays of an angle). By taking the one-sided Laplace transform of (*) over the variable t, (*) is reduced to a partial differential equation in r and θ. This reduced equation may be solved with the aid of the Green's function and $u(r, \theta, t)$ is then recovered from its Laplace transform by the inversion integral. The solution of the above problem is the sum of three expansions containing Bessel functions of r and sines and cosines of multiples of $\pi\theta/\theta_0$. Further types of boundary conditions are considered. *A. E. Heins.*

Koschmieder, Lothar. Die endliche Fouriersche Abbildung und ihr Nutzen bei Aufgaben der Wärmeleitung in Stäben und Platten. Deutsche Math. **5**, 521–545 (1941). [MF 4802]

The finite Fourier transform is used first to derive the formula for the temperatures in a homogeneous slab when the flux of heat is prescribed on the two bounding planes, and also when the flux is prescribed on one plane and the temperature on the other. From the transformation of the convolution of two functions the results are written in terms of Green's functions which are simple combinations of the theta functions. In a corresponding manner the double Fourier transformation is used to find the temperatures in a homogeneous rectangular plate with insulated faces, in the various cases arising when the flux of heat or the temperature is prescribed along the four edges. These results too are expressed in terms of the theta functions.

R. V. Churchill (Madison, Wis.).

Tranter, C. J. Note on a problem in the conduction of heat. Philos. Mag. (7) **31**, 432 (1941). [MF 4701]

The author points out that his solution of a problem in the conduction of heat [Philos. Mag. (7) **28**, 579–583 (1939); cf. these Rev. **1**, 181] will agree with R. V. Churchill's [Philos. Mag. (7) **31**, 81–87 (1941); cf. these Rev. **2**, 204], if in the expressions for v_2 of Tranter one replaces the misprint $(2n+1)a$ by $2(n+1)a$. *F. G. Dressel.*

Cinquini-Cibrario, Maria. Un complemento allo studio del problema di Dirichlet in dominii infiniti. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **76**, 105–124 (1941). [MF 4795]

The author treats certain boundary value problems concerning the equations

$$x^4\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2x^3\frac{\partial u}{\partial x} = 0$$

and

$$x^4\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

The treatment of these problems in the present paper is simpler than the one given by the author [Ann. Mat. Pura Appl. (4) **14**, 215–247 (1936)]. The reason for dealing with the above equations is the fact that Dirichlet problems for certain infinite domains can be reduced to problems of these equations for finite domains. *E. Rothe.*

Lewis, T. On the solution of two-dimensional problems of the Dirichlet and Neumann type. Quart. J. Math., Oxford Ser. **12**, 30–32 (1941). [MF 4669]

The author shows how to write down the complex potential for two-dimensional flow for an arbitrary cylinder when the stream function is prescribed on the cylinder and vanishes at infinity. By conformal mapping this is reduced to the problem of a circular cylinder, and an integral formula similar to Cauchy's integral formula is obtained. As illustrations of the method he obtains the potentials of a cylinder in the presence of a line vortex and of a doublet.

J. W. Green (Rochester, N. Y.).

Leonov, M. J. Problems and applications of the theory of potential. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] (N.S.) **4**, no. 5–6, 73–86 (1940). (Russian. English summary) [MF 4653]

The work deals with questions arising from problems of the theory of potential for cases in which a function, harmonic in space, or its normal derivative is given on the plane or on its part. The results obtained are applied to the solution of problems in electrostatics, hydrodynamics and the theory of elasticity. *Author's summary.*

Beckenbach, E. F. Functions having subharmonic logarithms. Duke Math. J. **8**, 393–400 (1941). [MF 4602]

A function $p(u, v)$ is said to be of class PL in a domain D if $p(u, v) \geqq 0$ and if $\log p(u, v)$ is subharmonic there. The

author presents several new characterizations of functions of class PL. Theorem: If $p(u, v) \geqq 0$ in D, then $p(u, v)$ is of class PL there if and only if $[(u-\alpha)^2+(v-\beta)^2]p(u, v)$ is subharmonic for every choice of the constants α, β. To state another result of the paper, let us designate by $A(f, Q, r)$, $L(f, Q, r)$ the integral means of a function $f(u, v)$ on the circular disc and on the perimeter of the circle with center Q and radius r, respectively. Theorem: If $p(u, v) \geqq 0$ is continuous in D, then $p(u, v)$ is of class PL there if and only if, for every continuous $q(u, v)$ of class PL, the inequality $A(pq, Q, r) \leqq L(p, Q, r)L(q, Q, r)$ holds for all circular discs in D. The proofs are similar to those used in previous characterizations of functions of class PL. *T. Radó.*

Saks, S. On the operators of Blaschke and Privaloff for subharmonic functions. Rec. Math. [Mat. Sbornik] N.S. **9** (**51**), 451–456 (1941). (English. Russian summary) [MF 4567]

Let $u(P)$ designate a function that is subharmonic in a domain G of xyz-space [all the results can be extended to n-space]. Let $S(Q, r)$ denote the sphere with center Q and radius r. If $S(Q, r)$ is comprised in G, then $m(u, Q, r)$ and $M(u, Q, r)$ will denote the integral means of u on the surface of $S(Q, r)$ and on $S(Q, r)$, respectively. Let us put, for brevity, $\gamma(u, Q, r) = [m(u, Q, r) - u(Q)]/(\frac{2}{3}\pi r^2)$, $\Gamma(u, Q, r) = [M(u, Q, r) - u(Q)]/(\frac{2}{3}\pi r^2)$. According to previous results of Blaschke and of Privaloff, we have $\overline{\lim}_{r \to 0} \gamma(u, Q, r) \geqq 0$, $\overline{\lim}_{r \to 0} \Gamma(u, Q, r) \geqq 0$ for every subharmonic function u. The author improves upon these results by showing that for every subharmonic function u the limits, for $r \to 0$, of $\gamma(u, Q, r)$ and of $\Gamma(u, Q, r)$ exist and are equal almost everywhere. Let $\sigma(E)$ be the non-negative mass distribution in terms of which u can be expressed, according to F. Riesz, as a potential plus a harmonic function. Then, quite precisely, the limits of $\gamma(u, Q, r)$ and of $\Gamma(u, Q, r)$ exist and are equal to the symmetric derivative of $\sigma(E)$ at every point where this derivative exists. The proof is based upon the representation, due to F. Riesz, of a subharmonic function by means of a non-negative mass distribution. *T. Radó* (Columbus, Ohio).

Privaloff, I. On a theorem of S. Saks. Rec. Math. [Mat. Sbornik] N.S. **9** (**51**), 457–460 (1941). (English. Russian summary) [MF 4568]

Using the notations of the preceding review, the limit inferior and the limit superior of $\Gamma(u, r, Q)$ are called the lower and the upper Laplace operators of u and are denoted by $\underline{\Delta}^* u$ and $\overline{\Delta}^* u$, respectively. Let $\underline{\rho}$, $\overline{\rho}$ denote the lower and the upper symmetric derivatives, respectively, of the mass distribution $\sigma(E)$. The author proves then that $\underline{\rho} \leqq \underline{\Delta}^* u \leqq \overline{\Delta}^* u \leqq \overline{\rho}$ everywhere in G, and obtains in this manner a new proof of the results of Saks described in the preceding review. *T. Radó* (Columbus, Ohio).

Privaloff, I. I. Sur la définition d'une fonction harmonique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 102–103 (1941). [MF 4693]

Using the notations of the preceding review, the results of this paper may be stated as follows. Lemma: If $u(Q)$ is continuous in G and if almost everywhere in G the condition $\overline{\Delta}^* u \geqq 0$ is satisfied, and if $\overline{\Delta}^* u > -\infty$ in G with the possible exception of a closed set of zero capacity, then u is subharmonic in G. From this lemma, the author derives the following theorem. Let u be continuous in G. Suppose that (1) $\underline{\Delta}^* u \geqq 0 \leqq \overline{\Delta}^* u$ almost everywhere in G; (2) $\overline{\Delta}^* u > -\infty$, $\underline{\Delta}^* u < -\infty$ in G with the possible exception of a closed set of zero capacity. Then u is harmonic in G. *T. Radó.*

Privaloff, I. I. Quelques applications de l'opérateur généralisé de Laplace. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 104–105 (1941). [MF 4694]

This paper contains restatements of and corollaries to the results stated in the preceding two reviews. For example: Let $u(Q)$ and $v(Q)$ be subharmonic in G. Using the notations of the preceding two reviews, suppose that (1) $\underline{\Delta}^* u < +\infty$, $\underline{\Delta}^* v < +\infty$ everywhere in G, and (2) $\underline{\Delta}^* u = \underline{\Delta}^* v$ almost everywhere in G. Then $u - v$ is harmonic in G. *T. Radó.*

NUMERICAL AND GRAPHICAL METHODS

***Table of Natural Logarithms. Vols. I, II.** Prepared by the Federal Works Agency, Work Projects Administration for the City of New York, as a Report of Official Project Nos. 765–97–3–10, 65–2–97–33; conducted under the sponsorship of the National Bureau of Standards. Technical Director: Arnold N. Lowan. New York, 1941. xviii+501 pp. each. $2.00 each.

This work contains the natural logarithms of the integers from 1 to 100000. The values are given to 16 decimals (up to 18 significant figures) and are printed in two columns. To follow are tables giving 16 decimal place values of the natural logarithms of the decimal numbers from 0 to 10 at intervals of .0001.

***Tables of Sine, Cosine and Exponential Integrals. Vol. II.** Prepared by the Federal Works Agency, Work Projects Administration for the City of New York, as a Report of Official Project No. 765–97–3–10; conducted under the sponsorship of the National Bureau of Standards. Technical Director: Arnold N. Lowan. New York, 1940. xxxvii+225 pp. $2.00.

Volume I was reviewed in these Rev. **2**, 239. In Volume II Table I (200 pages) gives the values of $\mathrm{Si}(x)$, $\mathrm{Ci}(x)$, $\mathrm{Ei}(x)$ and $-\mathrm{Ei}(-x)$ from $x=0$ to $x=9.999$ at intervals of 0.001, $\mathrm{Si}(x)$ and $\mathrm{Ci}(x)$ to 10 decimal places, $\mathrm{Ei}(x)$ to 10 and $-\mathrm{Ei}(-x)$ to 9 significant figures. Second central differences are tabulated. Table II (12 pages) gives values of $\mathrm{Si}(k\pi \pm h)$, $k=1, 2, 3$, and of $\mathrm{Ci}(k\pi/2 \pm h)$, $k=1, 3, 5$, to 15 decimal places for h from 0 to 0.01 at intervals of 0.0001 and from 0.01 to 0.05 at intervals of 0.001. Table III gives $\mathrm{Si}(x)$ and $\mathrm{Ci}(x)$ to 10 decimal places from $x=10$ to $x=20$ at intervals of 0.1 and from $x=20$ to $x=40$ at intervals of 0.2. Also the values of $\mathrm{Ei}(x)$ and $-\mathrm{Ei}(-x)$ from 10 to 15 at intervals of 0.1 to 10 significant figures for $\mathrm{Ei}(x)$ and to 14 decimals for $-\mathrm{Ei}(x)$. Tables IV and V give interpolating coefficients and Table VI multiples of $\pi/2$ and other constants. Volumes I and II are not mutually exclusive, but in part cover the same range (0 to 2) with different intervals. *W. E. Milne* (Corvallis, Ore.).

***Dwight, Herbert Bristol. Mathematical Tables.** McGraw-Hill Book Company, Inc., New York, 1941. vii+231 pp. $2.50.

This handy collection of tables is reproduced by a photo-offset method in small but very clear types. Most tables are to five significant figures with differences indicated.

Content: (i) Trigonometric functions and their logarithms in hundredths of degrees; sin, cos, tan of thousandths of radians; $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$; (ii) log nat x for $1<x<3$ in steps of .01, for $3<x<10$ in steps of .1, and log nat 10^k for integer $k<40$; e^x for $-5<x<5$, the hyperbolic functions for $0<x<3$ and their inverses, all in steps of .01; (iii) binomial coefficients up to 25; factorials or their logarithms up to 250; (iv) Gregory-Newton interpolation coefficients ${}_pC_n$ to 7 places for $n<6$, p in steps of .01; Lagrangean interpolation coefficients; $(1+x^2)^{\frac{1}{2}}$ for $0<x<1$ in steps of .01; (v) zonal harmonics $P_n(x)$ for $n\leqq 10$, in steps of .01; $P_n(\cos\theta)$ for $n\leqq 10$ and $dP_n(\cos\theta)/d\theta$ for $n\leqq 7$ in steps of $1'$; (vi) complete elliptic integral of first kind for $k^2<.96$ in steps of .01, for $.96<k^2<.997$ in steps of .001, for $.997<k^2<.9995$ in steps of .0001 and then in steps of .00001; complete elliptic integral of second kind in steps of .01; (vii) Bernoulli's and Euler's numbers and their logarithms for $n<35$; (viii) the gamma function for $1<x<2$, the error function for $0<x<2$, both in steps of .001; (ix) the Bessel function ber x, bei x, their derivatives, as well as the absolute value and the argument of ber $x+i$ bei x for $0<x<20$ in steps of .1; (x) $\zeta(s)$ for $-24<s<24$ in steps of .1 to 10 places; $(s-1)\zeta(s)$ for $-2.0<s<7$; finally a short table of common logarithms. *W. Feller* (Providence, R. I.).

Hohenberg, Fritz. Annäherung von Kurvenbögen durch Kreisbögen. Akad. Wiss. Wien, S.-B. IIa. **149**, 145–156 (1940). [MF 4938]

The circle of curvature is in general not the "best" approximation to the graph of a curve. If one wishes to replace a finite arc of the curve by a circular arc, as is usual in practice, a definition of best approximation in this sense is given, and the corresponding circles are constructed. *O. Neugebauer* (Providence, R. I.).

Friauf, James B. Nomograms for the solution of spherical triangles. J. Franklin Inst. **232**, 151–174 (1941). [MF 5007]

Nomograms are given for the laws of sines and cosines; also certain supplementary nomograms for use in cases where the basic ones are inaccurate. Applications are made to navigation and astronomy. *P. W. Ketchum.*

Nikoulin, N. A. An instrument for drawing a curve of the third order with a double point. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] (N.S.) **4**, no. 4, 121–124 (1940). (Russian. English summary) [MF 4649]

Willers, Fr. A. Benutzung projektiver Skalen zur Unterteilung von Skalen anderer Funktionen. Z. Angew. Math. Mech. **20**, 291–292 (1940).

There is obtained an expression for the remainder term when one interpolates to a given function by a linear fractional function. *P. W. Ketchum* (Urbana, Ill.).

Kneissl, M. Näherungsverfahren zum Zusammenschluss von Dreiecksnetzen. Allg. Vermessgs.-Nachr. **53**, 121–131, 169–174 (1941). [MF 4578]

Dwyer, P. S. The evaluation of determinants. Psychometrika **6**, 191–204 (1941). [MF 5094]

Dwyer, P. S. The solution of simultaneous equations. Psychometrika **6**, 101–129 (1941). [MF 4131]

This paper is essentially a nicely arranged review of various established techniques or rituals for the rapid solution of simultaneous numerical linear equations, some well adapted for pencil computation and others, principally the "Abbreviated Doolittle Method," when a computing machine is available. Ten more or less different techniques are discussed by applying each to the same example. Special emphasis is laid on short-cuts possible in the symmetric case: $\sum_{j=1}^{n} a_{ij}x_j = b_i$, where $a_{ij}=a_{ji}$. An extensive bibliography is appended. *A. L. Foster* (Berkeley, Calif.).

Ostrowski, Alexandre. Sur la convergence et l'estimation des erreurs dans quelques procédés de résolution des équations numériques. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 213–234. [MF 3514]

This paper offers a complete study of the conditions of validity and rapidity of convergence of (1) a method by iteration due to von Mises and H. Geiringer [Z. Angew. Math. Mech. **9**, 58–62 (1929)], (2) Newton's method for both the real and complex domain, (3) a less rapidly convergent simplification of Newton's method, proposed by the author, expressed by the relations $x_1=x_0-f(x_0)/f'(x_0)$, $x_2=x_1-f(x_1)/f'(x_0)$, $x_3=x_2-f(x_2)/f'(x_2)$, $x_4=x_3-f(x_3)/f'(x_2)$, $\cdots$, the denominator $f'(x_i)$ of the fractions being changed at every second step only. There are detailed references to previous work on Newton's method by Fourier, Cauchy, Schröder and Faber, their results being both simplified and sharpened. An attempt is made to compare these methods of solution with respect to the necessary computational labor measured in terms of a unit of labor called a "Horner" (=the labor of computing $f(x)$ or $f'(x)$ for a given x). In spite of the variability of this unit, the comparison indicates the greater economy of the author's alternating method (3), as compared with Newton's method, in reaching a certain given degree of exactness. *I. J. Schoenberg.*

Lidstone, G. J. Notes on interpolation. Part 2. III(i) continued. The origin of the throw-back device. IV. Aitken's new method of inverse interpolation. V. The connexion of the throw-back with Stirling's and Bessel's formulae. J. Inst. Actuar. **71**, 68–95 (1941). [MF 4140]

[The first part appeared in the same J. **68**, 267–296 (1938).] This paper is composed of a series of commentaries, illustrated frequently with numerical examples, on the subject of interpolation. In particular the author considers the so-called "throw-back device," namely the adjustment of lower order differences (as, for example, in the tables published by the British Association for the Advancement of Science) so as to compensate in part for the omission of higher order differences. He further considers at some length Aitken's method of inverse interpolation by quadratic cross-means. [Cf. Proc. Roy. Soc. Edinburgh **58**, 161–175 (1938).] *W. E. Milne* (Corvallis, Ore.).

Hoogland, J. J. Note on the numerical calculation of the orthogonal polynomials. Ann. Eugenics **11**, 77–79 (1941). [MF 4864]

A numerical method is presented for the computation of coefficients needed in fitting polynomial regression lines of any degree according to the technique described in § 27ff. of R. A. Fisher, Statistical Methods for Research Workers, 7th ed., 1938. *Z. W. Birnbaum* (Seattle, Wash.).

Okaya, Tokiharu. Numerical integration by Tchebychef's q-functions. Proc. Phys.-Math. Soc. Japan (3) **23**, 273–282 (1941). [MF 4768]

The author approximates the integrand over the interval of integration by means of orthogonal polynomials with

unit weight function. The integral of the polynomial approximation provides an approximation to the integral. The method is applied to the evaluation of an integral occurring in a hydrodynamical problem. *W. E. Milne.*

***Courant, R. On a method for the solution of boundary-value problems.** Theodore von Kármán Anniversary Volume, pp. 189–194. California Institute of Technology, Pasadena, Calif., 1941.

Hadamard's direct method for solving variational problems is discussed and applications to conformal mapping are given. This method was proposed by Hadamard in 1908 [Mémoire sur le problème d'analyse relatif a l'équilibre des plaques élastiques encastrées, Mémoires présentés par divers savants à l'Académie des Sciences, Paris, 1908] but has not received the same full attention that the Rayleigh-Ritz method has been given because of particular difficulties in the convergence problem. It has the advantage that it does not make use of "coordinate functions" which are required by the Rayleigh-Ritz method. It is remarked that in the light of recent developments in the theory of conformal mapping and the Plateau problem there is some hope that a full mathematical treatment of Hadamard's method may be expected. *A. E. Heins* (Lafayette, Ind.).

***Shortley, George H., Weller, Royal and Fried, Bernard. Numerical Solution of Laplace's and Poisson's Equations with Applications to Photoelasticity and Torsion.** Ohio State University Studies, Engineering Series, v. 11, no. 5. Engineering Experiment Station, Bulletin no. 107. Columbus, Ohio, 1940. iii+51 pp. $.60.

This pamphlet has as its main purpose a thorough and critical discussion of the technique and devices useful in applying the method of finite differences to the numerical solution of two dimensional boundary value problems. The authors confine their discussion to the Laplace equation, but the methods they advocate would also be useful for a variety of two dimensional boundary value problems. The general procedure recommended is a refinement of the iteration method ascribed to Liebmann, which consists in assuming the function values at the interior net points of the region and then improving these values by replacing the value at any interior net point with the average of the values at its four neighboring points. The authors recommend an iteration procedure which involves improvement of the function values using blocks of four, nine, or twenty-five net points. Investigation of the rapidity of convergence of the Liebmann and the "block" procedures indicates that the nine-block procedure is the best; the authors estimate that the nine-block procedure will yield numerical results of a given accuracy in less than one-third of the time required for the Liebmann procedure. After a few iterations, it is possible to extrapolate, then carry on iterations again, etc. Two problems from elasticity are treated in detail, with emphasis on a number of useful devices in addition to those already mentioned. *J. J. Stoker* (New York, N. Y.).

Frocht, M. M. and Leven, M. M. A rational approach to the numerical solution of Laplace's equation. J. Appl. Phys. 12, 596–604 (1941). [MF 4887]

If ψ_a, ψ_b, ψ_c and ψ_d denote the known values of a harmonic function $\psi(x, y)$ at the four points $A(=(a, 0))$, $B(=(0, b))$, $C(=(-c, 0))$, $D(=(0, -d))$, then a first approximation to the value of ψ at $O(=(0, 0))$ is given by

$$(1)\quad \left(\frac{1}{ac}+\frac{1}{bd}\right)\psi_0=\frac{1}{a(a+c)}\psi_a+\frac{1}{b(b+a)}\psi_b+\frac{1}{c(c+a)}\psi_c+\frac{1}{d(d+b)}\psi_d.$$

Based on this formula the authors present a procedure for the numerical solution of Laplace's equation with given boundary values. (1) Formula (1) is applied at suitable chosen key points O in the region, the points A, B, C, D being on the boundary. For each key point O the coordinate system is rotated about O through several angles up to $\pi/2$, and the corresponding values of ψ at O given by (1) are plotted as function of the angle. The average ψ for angles from 0 to $\pi/2$ is taken as the first approximation at the key point. (2) Additional points of a network are introduced and corresponding values of ψ are obtained by (1). (3) The values so obtained together with the key values are refined by the process of iteration in the usual manner. (4) The authors explain the modification to be made where nets of different size mesh adjoin, and also when curvilinear instead of rectangular coordinates are employed. Two examples based on problems in photoelasticity are worked out, one in rectangular and one in polar coordinates.

W. E. Milne (Corvallis, Ore.).

Mikeladze, Sch. Numerische Integration der Gleichungen vom elliptischen und parabolischen Typus. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 5, 57–74 (1941). (Russian. German summary) [MF 4270]

To solve approximately the equation

$$a\frac{\partial^2 u}{\partial x^2}+c\frac{\partial^2 u}{\partial y^2}+d\frac{\partial u}{\partial x}+e\frac{\partial u}{\partial y}-gu=l,$$

where a, c, d, e, g and l are continuous functions of x and y in a region A bounded by a closed curve γ, with assigned values of u on γ, the author spreads a lattice with square mesh of side h over A. The derivatives in the differential equation are replaced by differences agreeing up to and including terms of order h^2. For lattice points distant less than h from intersections of the lattice with γ special difference formulas are provided. There result n linear equations for the values of u at the n lattice points within A. These are solved by successive approximations. The author considers the magnitude of the error, solves a numerical example, extends the method to three dimensions and solves a problem in the conduction of heat. *W. E. Milne.*

Schlaefke, K. Zur harmonischen Analyse von Nockenkurven. Luftfahrtforschung 17, 87–88 (1940). [MF 3958]

The design of valve springs for high speed motors requires the harmonic analysis of the function $f(x)$ representing the valve opening for each angle x through which the cam has turned. The function $f(x)$ is even, has a continuous first derivative and is usually a composite of polynomials over subintervals of $(0, 2\pi)$. In view of these features, the formula

$$\int f(x)\cos \mu x\,dx=(\sin \mu x/\mu)f(x)+(\cos \mu x/\mu^2)f'(x)-(\sin \mu x/\mu^3)f''(x)-\cdots$$

provides a simple harmonic analysis of $f(x)$. The amplitudes of the harmonics are found by this method from the values of the derivatives of $f(x)$ at the ends of the subintervals. An example shows the simplicity of the computation.

R. V. Churchill (Madison, Wis.).

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